Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Systems & Control Letters 52 (2004) 137 – 147
SYSTEMS & CONTROL LETTERS
www.elsevier.com/locate/sysconle

On tracking and disturbance rejection by adaptive control

A. Ilchmann[a,*], E.P. Ryan[b]

[a]Institute of Mathematics, Technical University Ilmenau, Weimarer Straße 25, 98684 Ilmenau, Germany
[b]Department of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, UK

Received 13 March 2003; received in revised form 17 November 2003; accepted 17 November 2003

Abstract

For linear minimum-phase relative-degree-one systems subject to disturbances and nonlinear perturbations, results pertaining to disturbance rejection and adaptive tracking of constant reference signals are presented and discussed in the context of related results in the literature.

Keywords: Adaptive control; Disturbance rejection; Tracking

1. Introduction

Let $L(n)$ denote the class of linear, n-dimensional, real, single-input, single-output, minimum-phase plants having relative degree one. Let $L^+(n) \subset L(n)$ denote the subclass of systems with positive high-frequency gain. It is well known (see, for example, the seminal work in [3,9,10]) that the following adaptive proportional output feedback

$$u(t) = -k(t)y(t), \quad \dot{k}(t) = y^2(t), \quad k(0) = k^0 \in \mathbb{R}$$

applied to any system $\Sigma \in L^+(n)$, with $n \in \mathbb{N}$, renders the plant zero state globally attractive and, moreover, the adaptive gain k converges to a finite limit.

The purpose of the present note is to show that the inclusion of certain nonlinear terms and a non-adaptive convolution component in the above feedback strategy ensures (a) asymptotic tracking of constant reference signals r, (b) rejection of disturbances d which are finite linear combinations of constant and sinusoidal functions, (c) robustness with respect to globally Lipschitz state-dependent perturbations f and polynomial output-dependent perturbations g of degree $\deg(g) \leqslant \gamma$, where $\gamma \geqslant 1$ is known, see Fig. 1.

Moreover, the quadratic gain adaptation law ($\dot{k} = y^2$) in the above strategy is generalized to a considerably wider class of adaptation laws which, for linear systems, encompasses adaptation laws ($\dot{k} = \psi(|y|)$) with *bounded* ψ. See [6] for a treatment of generalized gain adaptation laws in a different context of controlled functional differential equations.

* Corresponding author. Tel.: +449-3677693623; fax: +49-3677693270.
E-mail addresses: ilchmann@mathematik.tu-ilmenau.de, achim.ilchmann@tu-ilmenau.de (A. Ilchmann), epr@maths.bath.ac.uk (E.P. Ryan).

0167-6911/$ - see front matter
doi:10.1016/j.sysconle.2003.11.007

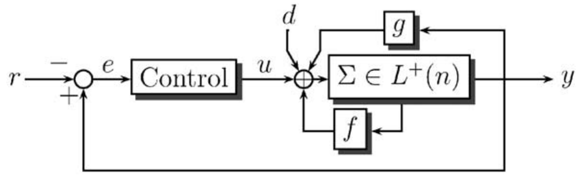

Fig. 1. Tracking control of perturbed systems.

1.1. System class

For $\gamma, m \in \mathbb{N}$ and $\omega = (\omega_1, \ldots, \omega_m) \in \mathbb{R}^m_{>0}$ (that is, a vector ω with positive components), define the system class $S(\gamma, m, \omega)$ to be the family of all finite-dimensional, linear, single-input, single-output, minimum-phase plants (A, b, c) having relative degree one and positive high-frequency gain $cb > 0$, and subject to a globally Lipschitz state-dependent perturbation f, a polynomial output-dependent perturbation $g \in \mathbb{R}[y]$ of degree $\deg(g) \leqslant \gamma$ and disturbance consisting of a finite linear combination of constant and sinusoidal functions:

$$\begin{cases} \dot{\xi}(t) = A\xi(t) + b[f(\xi(t)) + g(y(t)) + d(t) + u(t)], \quad \xi(0) = \xi^0, \\ y(t) = c\xi(t), \quad cb > 0, \\ \det \begin{bmatrix} sI_n - A & b \\ c & 0 \end{bmatrix} \neq 0 \quad \text{for all } s \in \mathbb{C} \quad \text{with } \operatorname{Re} s \geqslant 0, \\ f : \mathbb{R}^n \to \mathbb{R} \text{ globally Lipschitz}, \quad g \in \mathbb{R}[y] \text{ with } \deg(g) \leqslant \gamma, \\ d : \mathbb{R} \to \mathbb{R}, \quad t \mapsto d_0 + \sum_{i=1}^{m} (\alpha_i \cos \omega_i t + \beta_i \sin \omega_i t), \quad d_0, \alpha_i, \beta_i \in \mathbb{R}. \end{cases} \tag{1}$$

1.2. Control objective

The objective is tracking, by the output of any system of class $S(\gamma, m, \omega)$, of any constant reference signal $r \in \mathbb{R}$. In particular, an (adaptive) output feedback control strategy is sought which depends only on the data $\gamma, m \in \mathbb{N}$, $\omega \in \mathbb{R}^m_{>0}$ and $r \in \mathbb{R}$ and which, when applied to any plant (1) of class $S(\gamma, m, \omega)$, ensures that all signals (internal and external) are bounded and $y(t) \to r$ as $t \to \infty$.

1.3. Control strategy

Firstly, we define two classes of locally Lipschitz functions, each parameterized by $\gamma \in \mathbb{N}$.

$$\Phi_\gamma := \Big\{ \varphi \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{>0} \mid \varphi \text{ locally Lipschitz and non-decreasing}, \\ \liminf_{s \to \infty} (s^{\gamma-1}/\varphi(s)) > 0, \exists \epsilon > 0 \colon \varphi(s) \geqslant \epsilon[1 + s^{\gamma-1}] \forall s \geqslant 0 \Big\}, \tag{2}$$

$$\Psi_\gamma := \Big\{ \psi : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0} \mid \psi \text{ locally Lipschitz}, \psi(s) = 0 \iff s = 0, \\ \liminf_{s \to \infty} (\psi(s)/s^{\gamma-1}) > 0, \exists \sigma > 0 : \psi(s) \leqslant \sigma s^{\gamma+1} \forall s \geqslant 0 \Big\}. \tag{3}$$

For future reference, we note that, for each $\varphi \in \Phi_\gamma$ and $\psi \in \Psi_\gamma$, there exist $\sigma, \epsilon > 0$ such that

$$\psi(s) \leqslant (\sigma/\epsilon)\varphi(s)s^2 \quad \text{for all } s \geqslant 0. \tag{4}$$

Let $\gamma, m \in \mathbb{N}$, $\omega \in \mathbb{R}^m_{>0}$, $r \in \mathbb{R}$, and $\varphi \in \Phi_\gamma$, $\psi \in \Psi_\gamma$. Denote the tracking error by $e(t) = y(t) - r$. The proposed strategy is a combination of a nonlinear error feedback with an adaptive gain, $k(t)\varphi(|e(t)|)e(t)$, and a non-adaptive convolution component:

$$\begin{cases} e(t) = y(t) - r, \\ u(t) = -k(t)\varphi(|e(t)|)e(t) - \eta_0 \displaystyle\int_0^t e(s)\,\mathrm{d}s - \sum_{i=1}^m \eta_i \int_0^t [\cos\omega_i(t-s)]e(s)\,\mathrm{d}s, \\ \dot{k}(t) = \psi(|e(t)|), \quad k(0) = k^0 \in \mathbb{R}, \end{cases} \tag{5}$$

where $\eta_i > 0$, $i = 0, \ldots, m$, are arbitrary positive constants.

We emphasize the flexibility of choice of the functions φ and ψ. For example, typical choices for $\varphi \in \Phi_\gamma$ and $\psi \in \Psi_\gamma$ are given by

$$\varphi(s) = \tfrac{1}{2}[1 + s^{\gamma-1}], \quad \psi(s) = \min\{|s|^{\gamma+1}, |s|^{\gamma-1}\}.$$

In particular, if $\gamma = 1$, then the functions $s \mapsto \varphi(s) = 1$ and $s \mapsto \psi(s) = \min\{s^2, 1\}$ are admissible and, in addition, if the disturbance is constant (i.e. $d(t) = d_0$ for all t), then the following PI control with gain adaptation only on the proportional term is feasible

$$u(t) = -k(t)e(t) - \eta_0 \int_0^t e(s)\,\mathrm{d}s, \quad \dot{k}(t) = \min\{e^2(t), 1\}, \quad k(0) = k^0 \in \mathbb{R}.$$

1.4. Discussion

Before proceeding to the main result, we contrast the above strategy (5) with related strategies in the literature. In the context of the linear system class $L(n)$, adaptive controls that ensure tracking (and disturbance rejection) of signals that correspond to bounded solutions of known linear time-invariant differential equations are well established (encompassing the constant reference signals, the periodic disturbance signals and the system class $L^+(n)$ of the present note): such controls are based on an internal model principle wherein the "dynamics" of the reference signal (and disturbance) are incorporated in the plant dynamics via an appropriately chosen pre-filter or pre-compensator (see, for example, [4] and references therein). These results are extended to a linear, multivariable, infinite-dimensional setting in [8]. From an "internal model" viewpoint, the convolution term in (5) may be interpreted as a compensator C (designed on the basis of an internal model principle), with transfer function given by

$$C(s) = \frac{\eta_0}{s} + \sum_{i=1}^m \frac{\eta_i s}{s^2 + \omega_i^2},$$

and having the tracking error signal e as input. Against this more general background, the main contribution of the present note (Theorem 1 below) is of modest content but is distinguishable from the above-cited results in two ways: robustness with respect to nonlinear perturbations f and g; the explicit nature and simplicity of the compensator C which has the tracking error e as input (and is non-adaptive, in contrast to the pre-filters or pre-compensators of [4] and [8] which have the signal ke as input, k being an adaptive parameter of a "high-gain" nature). The merits and demerits of (5) vis-à-vis other existing alternatives and related results in the literature will be discussed further in Section 3 below.

2. The main result

Theorem 1. *Let $\gamma, m \in \mathbb{N}$, $\omega \in \mathbb{R}^m_{>0}$, $\varphi \in \Phi_\gamma$ and $\psi \in \Psi_\gamma$. For every $r \in \mathbb{R}$, $\xi^0 \in \mathbb{R}^n$ and $k^0 \in \mathbb{R}$, the application of the adaptive feedback (5) to any plant (1) of class $S(\gamma, m, \omega)$ yields a closed-loop initial-value problem which has unique maximal solution $(\xi, e, k) : [0, \tau) \to \mathbb{R}^{n+2}$. Moreover,*

(i) $\tau = \infty$,
(ii) $\lim_{t\to\infty}(\xi(t), e(t)) = (0, 0)$,
(iii) $\lim_{t\to\infty} k(t)$ *exists and is finite*,
(iv) $\lim_{t\to\infty}(u(t) + \sum_{i=1}^m (\alpha_i \cos w_i t + \beta_i \sin w_i t))$ *exists and is finite.*

Proof. Let ξ^0, k^0 be arbitrary.

Step 1: Firstly, we express the closed-loop initial-value problem (1), (5) in a convenient form. Since $cb > 0$ in (1), we have $\mathbb{R}^n = \ker c \oplus \operatorname{im} b$ and hence, under an appropriate coordinate transformation $\xi \mapsto (z, y)$, we may express plant (1) in the form

$$\begin{cases} \dot z(t) = A_1 z(t) + A_2 y(t), \quad z(0) = z^0, \\ \dot y(t) = A_3 z(t) + A_4 y(t) + cb[f(z(t),\ y(t)) + g(y(t)) + d(t) + u(t)], \quad y(0) = y^0 \end{cases} \tag{6}$$

for real matrices $A_1, \ldots, A_4$ of conforming formats, $((z^0)^{\mathrm{T}},\ (y^0)^{\mathrm{T}})^{\mathrm{T}} = \xi^0$, and where the minimum phase property of (1) implies that A_1 is Hurwitz.

Define constants

$$\rho := -A_1^{-1} A_2 r, \quad p_0 := A_3 \rho + A_4 r + cb[d_0 + f(\rho, r) + g(r)]$$

and functions

$$p : \mathbb{R} \to \mathbb{R}, \quad t \mapsto d(t) + f(\rho, r) + g(r) + [A_3 \rho + A_4 r]/cb,$$

$$f^r : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}, \quad (x, e) \mapsto f(x + \rho,\ e + r) - f(\rho, r), \quad g^r : \mathbb{R} \to \mathbb{R}, \quad e \mapsto g(e + r) - g(r).$$

Evidently, f^r is globally Lipschitz with $f^r(0,0)=0$ and g^r is polynomial of degree γ with $g^r(0)=0$. Introducing the variables

$$x(t) := z(t) - \rho, \quad e(t) := y(t) - r,$$

and writing

$$v_0(t) := p_0 - \eta_0 \int_0^t e(s)\,\mathrm{d}s,$$

$$v_i(t) := \alpha_i \cos \omega_i t + \beta_i \sin \omega_i t - \eta_i \int_0^t [\cos \omega_i (t - s)] e(s)\,\mathrm{d}s, \quad i = 1, \ldots, m,$$

$$w_i(t) := \dot v_i(t) + \eta_i e(t), \quad i = 1, \ldots, m$$

the closed-loop initial-value problem (1), (5) may be expressed in the form

$$\left\{\begin{array}{l} \dot x(t) = A_1x(t) + A_2e(t), \quad x(0) = z^0 - \rho, \\ \dot e(t) = A_3x(t) + A_4e(t) + cb[f^r(x(t), e(t)) + g^r(e(t))] \\ \qquad\qquad + cb\displaystyle\sum_{i=0}^{m} v_i(t) - cbk(t)\varphi(|e(t)|)e(t), \quad e(0) = y^0 - r, \\ \dot v_0(t) = -\eta_0 e(t), \quad v_0(0) = p_0, \\ \begin{bmatrix} \dot v_i(t) \\ \dot w_i(t) \end{bmatrix} = \begin{bmatrix} w_i(t) - \eta_i e(t) \\ -\omega_i^2 v_i(t) \end{bmatrix}, \quad \begin{bmatrix} v_i(0) \\ w_i(0) \end{bmatrix} = \begin{bmatrix} \alpha_i \\ \alpha_i - \eta_i[y^0 - r] \end{bmatrix}, \quad i = 1, \ldots, m, \\ \dot k(t) = \psi(|e(t)|), \quad k(0) = k^0. \end{array}\right. \tag{7}$$

Noting that the functions on the right-hand sides of the above system of differential equations are locally Lipschitz, it follows that the initial-value problem (7) has a unique maximal solution $(x, e, v_0, (v_1, w_1), \ldots, (v_m, w_m), k) : [0, \tau) \to \mathbb{R}^{n+2(m+1)+1}$, for some $\tau \in (0, \infty]$.

Introducing the matrices $\hat A \in \mathbb{R}^{2(m+1)\times 2(m+1)}$ and $\hat B \in \mathbb{R}^{2(m+1)}$ given by

$$\hat A := \begin{bmatrix} \begin{bmatrix} -1 & cb \\ -\eta_0 & 0 \end{bmatrix} & \begin{bmatrix} cb & 0 \\ 0 & 0 \end{bmatrix} & \cdots & \begin{bmatrix} cb & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} -\eta_1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ -\omega_1^2 & 0 \end{bmatrix} & & 0 \\ \vdots & & \ddots & \\ \begin{bmatrix} -\eta_m & 0 \\ 0 & 0 \end{bmatrix} & 0 & & \begin{bmatrix} 0 & 1 \\ -\omega_m^2 & 0 \end{bmatrix} \end{bmatrix}, \quad \hat B := \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

and defining $q : [0, \tau) \to \mathbb{R}^{2(m+1)}$ and $\theta : [0, \tau) \to \mathbb{R}$ by

$$q(t) := [e(t),\ v_0(t),\ v_1(t),\ w_1(t), \ldots, v_m(t),\ w_m(t)]^{\mathrm{T}}, \tag{8}$$

$$\theta(t) := A_3x(t) + A_4e(t) + cb[f^r(x(t), e(t)) + g^r(e(t))] - cbk(t)\varphi(|e(t)|)e(t) + e(t), \tag{9}$$

the system of differential equations in (7) may be written as

$$\left\{\begin{array}{l} \dot x(t) = A_1x(t) + A_2e(t), \quad x(0) = x^0, \\ \dot q(t) = \hat A q(t) + \hat B\theta(t), \quad q(0) = q^0, \\ \dot k(t) = \psi(|e(t)|), \quad k(0) = k^0. \end{array}\right. \tag{10}$$

Step 2: Next, we show that $\hat A$ is Hurwitz. A straightforward (but tedious) calculation shows that $\hat A$ has characteristic polynomial given by

$$\pi(s) := [cb\,\eta_0 + (1+s)s]\prod_{j=1}^{m}(s^2 + w_j^2) + \sum_{j=1}^{m} cb\eta_j \prod_{\substack{k=1 \\ k\neq j}}^{m}(s^2 + w_k^2).$$

Noting that $\pi(i\beta) \neq 0$ for all $\beta \in \mathbb{R}$, we may write

$$\pi(s) = \frac{\tilde\pi(s)}{s\prod_{j=1}^{m}(s^2 + \omega_j^2)},$$

where

$$\tilde{\pi}(s) := 1 + s + cb\left[\frac{\eta_0}{s} + \sum_{j=1}^{m}\frac{\eta_i s}{s^2+\omega_j^2}\right] \quad \text{for all } s\in\mathbb{C} \text{ with } \mathrm{Re}(s) > 0.$$

Observe that, for all $s\in\mathbb{C}$ with $\mathrm{Re}(s)>0$,

$$\mathrm{Re}(\tilde{\pi}(s)) = 1 + cb\,\mathrm{Re}(s)\left[\frac{1}{cb} + \frac{\eta_0}{|s|^2} + \sum_{j=1}^{m}\frac{\eta_j(\omega_j^2+|s|^2)}{|s^2+\omega_j^2|^2}\right] > 0.$$

We may now conclude that the polynomial π has no roots in the closed right-half complex plane $\{s\in\mathbb{C} \mid \mathrm{Re}(s) \geqslant 0\}$ and so the matrix $\hat{A}$ is Hurwitz.

Step 3: We highlight some consequent inequalities. Since $\dot{q} = \hat{A}q + \hat{B}\theta$ is an exponentially stable linear system with input θ, there exists $c_0 = c_0(\hat{A},\hat{B},q^0) > 0$ such that

$$|e(t)| \leqslant \|q(t)\| \leqslant c_0\left[1 + \max_{s\in[0,t]}|\theta(s)|\right] \quad \forall t\in[0,\tau). \tag{11}$$

By properties of $\varphi\in\Phi_\gamma$, and noting that g^r is polynomial of degree γ with $g^r(0)=0$, we may infer the existence of constants $\epsilon,\mu>0$ such that

$$|g^r(e)| \leqslant \mu[1+|e|^{\gamma-1}]|e| \leqslant (\mu/\epsilon)\varphi(|e|)|e| \quad \forall e\in\mathbb{R}. \tag{12}$$

By (9), together with the global Lipschitz property of f^r (with $f^r(0,0)=0$) and property (12) of g^r, there exists a constant $c_1>0$ such that

$$|\theta(t)| \leqslant c_1[\|x(t)\| + |e(t)| + (1+|k(t)|)\varphi(|e(t))|e(t)|] \;\; \forall t\in[0,\tau). \tag{13}$$

Since $\dot{x} = A_1x + A_2e$, with A_1 Hurwitz, there exists a constant $c_2>0$ such that

$$\|x(t)\| \leqslant c_2\left[1 + \max_{s\in[0,t]}|e(s)|\right] \quad \forall t\in[0,\tau). \tag{14}$$

Combining (11), (13) and (14), we may infer the existence of $c_3>0$ such that

$$\max_{s\in[0,t]}|\theta(s)| \leqslant c_3\left[1 + \max_{s\in[0,t]}[(1+|k(s)|)\varphi(|e(s)|)|e(s)|]\right] \quad \forall t\in[0,\tau), \tag{15}$$

whence, invoking (8), (10) and (11), the existence of constants $c_4,c_5>0$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|e(t)| &\leqslant |\dot{e}(t)| \leqslant \|\dot{q}(t)\| \leqslant c_4[\|q(t)\| + |\theta(t)|]\\ &\leqslant c_5\left[1 + \max_{s\in[0,t]}[(1+|k(s)|)\varphi(|e(s)|)|e(s)|]\right] \quad \forall t\in[0,\tau).\end{aligned} \tag{16}$$

We will exploit the above inequality in due course.

Since A_1 is Hurwitz, there exists symmetric positive-definite $P\in\mathbb{R}^{(n-1)\times(n-1)}$ such that $PA_1 + A_1^{\mathrm{T}}P + I = 0$. Consider the C^1-function V: $[0,\tau)\to\mathbb{R}$ given by

$$V(t) := \frac{1}{2}\left[x(t)^{\mathrm{T}}Px(t) + e^2(t) + \frac{v_0^2(t)}{\eta_0} + \sum_{i=1}^{m}\frac{v_i^2(t) + (w_i(t)/\omega_i)^2}{\eta_i}\right] + k(t)$$

with derivative

$$\begin{aligned}\dot{V}(t) = {}& -\tfrac{1}{2}\|x(t)\|^2 + [A_2^{\mathrm{T}}P + A_3]x(t)e(t) + cbf^r(x(t),e(t))e(t) + A_4e^2(t)\\ &+ cbg^r(e(t))e(t) - cb\varphi(|e(t)|)e^2(t) + \psi(|e(t)|) \quad \forall t\in[0,\tau).\end{aligned}$$

Invoking the Lipschitz property of f^r (with Lipschitz constant λ) and defining

$$c_6 := \tfrac{1}{2}[\|A_2^{\mathrm{T}}P + A_3\| + cb\lambda]^2 + |A_4| + cb\lambda,$$

then elementary estimates yield

$$\dot V(t) \leqslant c_6 e^2(t) + cbg^r(e(t))e(t) - cb\varphi(|e(t)|)e^2(t) + \psi(|e(t)|) \quad \forall t \in [0,\tau).$$

By (4), together with property (12) of g^r and setting $c_7 := (cb\mu + c_6 + \sigma)/\epsilon$, we have

$$\dot V(t) \leqslant [c_7 - cbk(t)]\varphi(|e(t)|)e^2(t) \quad \forall t \in [0,\tau). \tag{17}$$

Step 4: We now establish boundedness of k and e on $[0,\tau)$.
Seeking a contradiction, suppose that the non-decreasing function k is unbounded. Then V is also unbounded. By (17), there exists $T \in [0,\tau)$ such that $\dot V(t) \leqslant 0$ for all $t \in [T,\tau)$ which contradicts unboundedness of V. Therefore, k is bounded on $[0,\tau)$.

Again seeking a contradiction, suppose that e is unbounded. For each $n \in \mathbb{N}$, define

$$\tau_n := \inf\{t \in [0,\tau) |\ |e(t)| = |e(0)| + n + 1\}, \quad \sigma_n := \sup\{t \in [0,\tau_n) |\ |e(t)| = |e(0)| + n\}.$$

Observe that, for all $n \in \mathbb{N}$,

$$\max_{s\in[0,t]} |e(s)| = \max_{s\in[\sigma_n,t]} |e(s)| \leqslant |e(0)| + n + 1 \leqslant 2(|e(0)| + n) \leqslant 2|e(t)| \quad \forall t \in [\sigma_n,\tau_n].$$

Furthermore, for all $n \in \mathbb{N}$ and all $t \in [\sigma_n,\tau_n]$, $|e(t)| \geqslant |e(0)| + n \geqslant 1$ and so, for all $n \in \mathbb{N}$ and all $t \in [\sigma_n,\tau_n]$,

$$1 + \max_{s\in[0,t]} \varphi(|e(s)|)|e(s)| \leqslant |e(t)|^{\gamma} + 2\varphi(2|e(t)|)|e(t)| \tag{18}$$

wherein we have appealed to monotonicity of φ. Recalling that $\liminf_{s\to\infty}(s^{\gamma-1}/\varphi(s)) > 0$, there exists $N_1 \in \mathbb{N}$ and $c_8 > 0$ such that

$$\varphi(2|e(t)|) \leqslant c_8|e(t)|^{\gamma-1} \quad \forall n \geqslant N_1 \ \forall t \in [\sigma_n,\tau_n],$$

which, together with (16) and (18) and boundedness of k, yields

$$\frac{\mathrm{d}}{\mathrm{d}t}|e(t)| \leqslant c_9|e(t)|^{\gamma} \quad \forall n \geqslant N_1 \ \forall t \in [\sigma_n,\tau_n],$$

for some constant $c_9 > 0$. Recalling that $\liminf_{s\to\infty}(\psi(s)/s^{\gamma-1}) > 0$, there exists $N_2 \in \mathbb{N}$ and $c_{10} > 0$ such that

$$\psi(|e(t)|) \geqslant c_{10}|e(t)|^{\gamma-1} \quad \forall n \geqslant N_2 \ \forall t \in [\sigma_n,\tau_n].$$

We may assume that $N_2 \geqslant N_1$. We may now conclude that

$$\begin{aligned}
\ln\left(\frac{n+1+|e(0)|}{N_2+1+|e(0)|}\right) &= \ln|e(\tau_n)| - \ln|e(\sigma_{N_2})| = \sum_{j=N_2}^{n}[\ln|e(\tau_j)| - \ln|e(\sigma_j)|] \\
&\leqslant \sum_{j=N_2}^{n}\int_{\sigma_j}^{\tau_j}\frac{(\mathrm{d}/\mathrm{d}t)|e(t)|}{|e(t)|}\,\mathrm{d}t \leqslant c_9\sum_{j=N_2}^{n}\int_{\sigma_j}^{\tau_j}|e(t)|^{\gamma-1}\,\mathrm{d}t \\
&\leqslant \frac{c_9}{c_{10}}\sum_{j=N_2}^{n}\int_{\sigma_j}^{\tau_j}\psi(|e(t)|)\,\mathrm{d}t = \frac{c_9}{c_{10}}\sum_{j=N_2}^{n}[k(\tau_j) - k(\sigma_j)] \\
&\leqslant \frac{c_9}{c_{10}}[k(\tau_n) - k(\sigma_{N_2})] \quad \forall n \geqslant N_2
\end{aligned}$$

which contradicts the fact that k is a bounded function. Therefore, e is bounded.

Step 5: Finally, we establish assertions (i)–(iv). By boundedness of e and k, together with (11), (14) and (15), it follows that the unique maximal solution $(x, e, v_0, (v_1, w_1), \ldots, (v_m, w_m))$ is bounded and so $\tau = \infty$. This establishes assertion (i). Since k is bounded with $\dot{k}(t) = \psi(|e(t)|)$ for all $t \in [0, \infty)$, we have $\psi(|e(\cdot)|) \in L^1[0, \infty)$. By boundedness of (e, k) and (16), $\dot{e}$ is bounded and so e is uniformly continuous. By continuity of ψ together with boundedness and uniform continuity of e, the function $\psi(|e(\cdot)|) \in L^1[0, \infty)$ is uniformly continuous. By Barbălat's lemma [1], it follows that $\psi(|e(t)|) \to 0$ as $t \to \infty$. By properties of the continuous function ψ, it follows that $e(t) \to 0$ as $t \to \infty$ which in turn, by the Hurwitz property of A_1 and the first of equations (10), implies that $x(t) \to 0$ as $t \to \infty$. This establishes Assertion (ii). Assertion (iii) is an immediate consequence of boundedness and monotonicity of k.

Assertions (i)–(iii) imply that the function θ, given by (9), is such that $\theta(t) \to 0$ as $t \to \infty$. Since $\hat{A}$ is Hurwitz, it follows from the second of equations (10) that $q(t) \to 0$ as $t \to \infty$ and so, by (8), $(v_0(t), (v_1(t), w_1(t)), \ldots, (v_m(t), w_m(t))) \to 0$ as $t \to \infty$. With $u_0 := d_0 + f(z^r, r) + g(r) + (cb)^{-1}[A_3 z^r + A_4 r]$ we see that

$$u_0 + u(t) + \sum_{i=1}^{m} (\alpha_i \cos w_i t + \beta_i \sin w_i t) = -k(t)\varphi(|e(t)|)e(t) + \sum_{i=0}^{m} v_i(t) \to 0$$

as $t \to \infty$, whence assertion (iv). This completes the proof. □

3. Remarks on related results

3.1. PI regulators for nonlinear systems

In a context of systems of the form

$$z^{(n)}(t) = F(z(t), \dot{z}(t), \ldots, z^{(n-1)}(t)) + bu(t), \quad b \geqslant b_0 > 0 \tag{19}$$

and assuming availability of the full state $\zeta(t) := (z(t), \dot{z}(t), \ldots, z^{(n-1)}(t))$ for feedback purposes, PI regulators have been studied in [2]. (To conform with the notation of the present note, we alter the nomenclature of [2].) The function F is assumed globally Lipschitz. We will show that variants of the results in [2] are easily obtained using ideas from the present note. Let $c = (c_1, \ldots, c_{n-1}, 1) \in \mathbb{R}^n$ be such that the polynomial $\pi(s) = c_1 + \cdots + c_{n-1}s^{n-1} + s^n$ does not have any root with non-negative real part. In [2], under the assumption that F is continuously differentiable it is shown that, for sufficiently large fixed gain $k > 0$, the following PI control (see [2, Eq. (7)]) ensures exponential convergence to zero of the state ζ:

$$u(t) = -k\left[\int_0^t \langle c, \zeta(s)\rangle \, \mathrm{d}s + \zeta_n(t) - \zeta_n(0)\right].$$

Regarding $y(t) = \langle c, \zeta(t)\rangle$ as an output from system (19) and defining

$$x(t) := (z(t), \dot{z}(t), \ldots, z^{n-2}(t)) \in \mathbb{R}^{n-1},$$

it is readily verified that system (19) may be expressed in the form

$$\dot{x}(t) = A_1 x(t) + A_2 y(t), \quad \dot{y}(t) = A_3 x(t) + A_4 y(t) + f(x(t), y(t)) + bu(t), \tag{20}$$

where $A_1 \in \mathbb{R}^{(n-1)\times(n-1)}$ has characteristic polynomial π (and so is Hurwitz) and $f : \mathbb{R}^n \to \mathbb{R}$ is the globally Lipschitz function (with Lipschitz constant $\lambda > 0$) given by

$$f : (x, y) \mapsto F\left(x, y - \sum_{i=1}^{n-1} c_i x_{i+1}\right).$$

Evidently, system (20) is of form (6) with $g=0$, $d=0$ and $b \geqslant b_0 > 0$ playing the rôle of cb. By Theorem 1 (with $r=0$), it immediately follows that the adaptive PI control (with arbitrary $\eta > 0$)

$$u(t) = -k(t)y(s) - \eta \int_0^t y(s)\,\mathrm{d}s, \quad \dot{k}(t) = y(t)^2, \quad k(0) = k^0 \in \mathbb{R}$$

ensures convergence to zero of the state of (19) for all initial data and convergence of the control u to $-F(0)/b$.

To obtain a variant of the non-adaptive result in [2, Theorem 4.1] (and without positing continuous differentiability of F), we seek a threshold value $k^* \geqslant 0$ such that, for all fixed $k > k^*$ and all $\eta > 0$, the non-adaptive PI control

$$u(t) = -ky(t) - \eta \int_0^t y(s)\,\mathrm{d}s \tag{21}$$

ensures global asymptotic stability of the closed-loop system and convergence of the control u to $-F(0)/b$. To this end, we first express the closed-loop system in the form of a non-adaptive analogue of (7):

$$\left\{\begin{aligned} &\dot{x}(t) = A_1 x(t) + A_2 y(t), \quad x(0) = x^0, \\ &\dot{y}(t) = A_3 x(t) + A_4 y(t) + f^0(x(t), y(t)) - bky(t) + v(t), \quad y(0) = y^0, \\ &\dot{v}(t) = -(\eta b)e(t), \quad v(0) = f(0,0), \end{aligned}\right. \tag{22}$$

where $f^0(x,y) := f(x,y) - f(0,0)$ and $v(t) := f(0,0) - \eta b \int_0^t y(s)\,\mathrm{d}s$. Since the right-hand side of the above system of equations is globally Lipschitz, for each $(x^0, y^0) \in \mathbb{R}^{n-1} \times \mathbb{R}$, the initial-value problem (22) has unique solution $(x,y) : [0,\infty) \to \mathbb{R}^N$. Let $\epsilon > 0$ be arbitrarily small and let $P = P^{\mathrm{T}} > 0$ be the unique solution of $PA_1 + A_1^{\mathrm{T}} P + (1+2\varepsilon)I = 0$. Then, writing

$$c_0 := [\|PA_2 + A_3\| + \lambda]^2 + |A_4| + \lambda,$$

the function $V : [0,\infty) \to [0,\infty)$, $t \mapsto \frac{1}{2}[\langle x(t), Px(t)\rangle + y(t)^2 + v(t)^2/(\eta b)]$, satisfies

$$\begin{aligned} \dot{V}(t) &= -\tfrac{1}{2}(1+2\epsilon)\|x(t)\|^2 + [A_2^{\mathrm{T}} P + A_3]x(t)y(t) + f^0(x(t),y(t))y(t) + [A_4 - bk]y(t)^2 \\ &\leqslant -\epsilon\|x(t)\|^2 - [c_0 - bk(t)]y(t)^2 \quad \forall t \in [0,\infty). \end{aligned}$$

Defining $k^* := c_0/b_0$ it immediately follows that, for each fixed $k > k^*$, the closed-loop system is stable. Moreover, an application of LaSalle's invariance principle yields global attractivity of the origin, whence global asymptotic stability of the closed-loop system (22). Since $(x(t), y(t), v(t)) \to (0,0,0)$ as $t \to \infty$, it follows from (20) and (22) that $u(t) \to -F(0)/b$.

In the special case of the scalar example $\dot{y}(t) = f(y(t)) + bu(t)$ with f globally Lipschitz (Lipschitz constant λ) and $b \geqslant b_0 > 0$ considered in [2, Section 4.2], the function $t \mapsto V(t) := \frac{1}{2}y(t)^2 + v(t)^2/(\eta b)$ has derivative $\dot{V}(t) = f^0(y(t))y(t) - bky(t)^2 \leqslant [\lambda - b_0 k]y(t)^2$ for all $t \in [0,\infty)$ which yields a threshold gain value of λ/b_0 (a significant improvement on the threshold value $9.124\lambda/b_0$ given in [2, Corollary 4.2]).

In the specific context of the second-order system considered in [2, Section 5.1], namely, $\ddot{w}(t) = 1 + \cos w(t) + u(t)$, and writing $y(t) = w(t) + \dot{w}(t)$, we have $A_1 = -1 = A_3$, $A_2 = 1 = A_4$, $\lambda = 1$ and $P = \varepsilon + \frac{1}{2}$. Chosing $\varepsilon = \frac{1}{2}$, then the above result implies that the control $u(t) = -ky(t) - \eta \int_0^t y(s)\,\mathrm{d}s$ ensures global asymptotic stability of the closed-loop system for all $\eta > 0$ and $k > k^* = 2$ (by exploiting the fact that, in this example, the nonlinearity depends only on $w(t)$, the latter threshold may be sharpened to $k^* = 1$). Again, this compares favorably with the result in [2] which, in the current notation, shows that the control $u(t) = -k[\int_0^t y(s)\,\mathrm{d}s + \dot{w}(t) - \dot{w}(0)]$ stabilizes the system for all $k > 25$.

3.2. Tracking by discontinuous feedback

Consider again a system of the form (1) belonging to $S(\gamma, m, \omega)$. This system is encompassed by the considerably more general framework of [12] and the approach therein applies to conclude that the adaptive discontinuous output feedback strategy (suitably interpreted in a set-valued sense)

$$u(t) = -k(t)[1 + |e(t)|^{\gamma-1}]e(t) - k(t)\,\mathrm{sgn}(e(t)),$$
$$\dot{k}(t) = |e(t)| + e(t)^2 + |e(t)|^{\gamma+1}, \quad k(0) = k^0 \in \mathbb{R} \tag{23}$$

ensures boundedness of all signals, convergence to zero of the tracking error and convergence to a finite limit of the gain function k. Here, the (non-adaptive) convolution term in (5) has been replaced by an adaptive discontinuous feedback term. This approach has additional benefits: the requisite tracking, disturbance rejection, boundedness of signals and convergence of gain properties persist when the constant reference signal r and the disturbance signal d (of the type considered in the present note) are replaced by any functions r and d of Sobolev class $W^{1,\infty}$ (absolutely continuous and bounded with essentially bounded derivative). The price paid in replacing (5) by (23) lies primarily in the discontinuous nature of the feedback: small measurement noise and/or inaccuracies in the implementation of $\mathrm{sgn}(e(t))$ may lead to errors with destabilizing effects. Moreover, stability analysis of the closed-loop system necessitates the adoption of an existence theory within framework of differential inclusions wherein uniqueness of solutions of the closed-loop initial value may not be assured.

3.3. Approximate tracking by continuous feedback

Let sat_λ and d_λ denote the (globally Lipschitz) functions $\mathbb{R} \to \mathbb{R}$ given by

$$\mathrm{sat}_\lambda(e) := \begin{cases} \mathrm{sgn}(e), & |e| \geqslant \lambda, \\ e, & |e| < \lambda, \end{cases} \qquad d_\lambda(e) := \max\{0, |e| - \lambda\}$$

and parametrized by $\lambda > 0$. Elaborating further the approach of Section 3.2, we remark that, if in place of the discontinuous feedback strategy, the following continuous feedback is employed

$$\begin{cases} u(t) = -k(t)[1 + |e(t)|^{\gamma-1}]e(t) - k(t)\mathrm{sat}_\lambda(e(t)), \\ \dot{k}(t) = [1 + |e(t)| + |e(t)|^{\gamma}]d_\lambda(e(t)), \quad k(0) = k^0 \in \mathbb{R}, \end{cases} \tag{24}$$

then the results of [7] (with precursors [5,11]), applied in the context of the plants belonging to $S(\gamma, m, \omega)$, ensure boundedness of all signals in the closed-loop system and convergence of the gain to a finite limit. Moreover, the Lipschitz nature of the right-hand sides of the differential equations governing the closed-loop behaviour ensure uniqueness of solutions for the initial-value problem. The price paid in replacing (5) by (24) is that convergence to zero of the tracking error is not assured: instead, convergence to the prescribed interval $[-\lambda, \lambda]$ is guaranteed, where $\lambda > 0$ may be chosen arbitrarily small. However, the boundedness and convergence properties persist when the constant reference signal r and the disturbance signal d (of the type considered in the present note) are replaced by any functions r and d of Sobolev class $W^{1,\infty}$. This added generality, together with the relative simplicity of the strategy and the fact that the accuracy quantifier $\lambda > 0$ may be chosen arbitrarily small, might argue for preference of (24) over (5).

References

[1] I. Barbălat, Systèmes d'équations différentielles d'oscillations nonlinéaires, Rev. Math. Pures Appl. Bucharest IV (1959) 267–270.
[2] A. Brockwell, A regulator for a class of unknown continuous-time nonlinear systems, Systems Control Lett. 44 (2001) 405–412.
[3] C.I. Byrnes, J.C. Willems, Adaptive stabilization of multivariable linear systems, Proceedings of the 23rd Conference on Decision and Control, Las Vegas, 1984, pp. 1574–1577.

[4] U. Helmke, D. Prätzel-Wolters, S. Schmid, Adaptive tracking for scalar minimum-phase systems, in: D. Hinrichsen, B. Mårtensson (Eds.), Control of Uncertain Systems, Birkhäuser, Boston, 1990, pp. 101–117.
[5] A. Ilchmann, E.P. Ryan, Universal λ-tracking for nonlinearly-perturbed systems in the presence of noise, Automatica 30 (1994) 337–346.
[6] A. Ilchmann, E.P. Ryan, On gain adaptation in adaptive control, IEEE Trans. Automat. Control 48 (2003) 895–899.
[7] A. Ilchmann, E.P. Ryan, C.J. Sangwin, Systems of controlled functional differential equations and adaptive tracking, SIAM J. Control Optim. 40 (2002) 1746–1764.
[8] H. Logemann, A. Ilchmann, An adaptive servomechanism for a class of infinite-dimensional systems, SIAM J. Control Optim. 32 (1994) 917–936.
[9] I. Mareels, A simple selftuning controller for stably invertible systems, Systems Control Lett. 4 (1984) 5–16.
[10] A.S. Morse, Recent problems in parameter adaptive control, in: I.D. Landau (Ed.), Outils et Modèles Mathématiques pour l'Automatique, l'Analyse de Systèmes et le Traitment du Signal, Editions du CNRS 3, Paris, 1983, pp. 733–740.
[11] E.P. Ryan, A nonlinear universal servomechanism, IEEE Trans. Automat. Control AC-39 (1994) 753–761.
[12] E.P. Ryan, An integral invariance principle for differential inclusions with application in adaptive control, SIAM J. Control Optim. 36 (1998) 960–980.