International Journal of Control
Vol. 79, No. 8, August 2006, 910–917

Asymptotic tracking with prescribed transient behaviour for linear systems

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(*Received 13 May 2005; revised 7 February 2006; accepted 21 March 2006*)

The problem of asymptotic tracking of reference signals is considered in the context of m-input, m-output linear systems (A, B, C) with the following structural properties: (i) CB is sign definite (but not necessarily symmetric), (ii) the zero dynamics are exponentially stable. The class $\mathcal{Y}_{\text{ref}}(\alpha)$ of reference signals is the set of all possible solutions of a fixed, stable, linear, homogeneous differential equation (with associated characteristic polynomial α). The first control objective is asymptotic tracking, by the system output $y = Cx$, of any reference signal $r \in \mathcal{Y}_{\text{ref}}(\alpha)$. The second objective is guaranteed error $e = y - r$ transient performance: e should evolve within a prescribed performance funnel $\mathcal{F}_\varphi$ (determined by a function φ). Both objectives are achieved simultaneously by an internal model in series with a proportional time-varying error feedback $t \mapsto u(t) = -\nu(k(t))e(t)$, where ν is a smooth function with the properties $\limsup_{\kappa\to\infty} \nu(\kappa) = +\infty$ and $\liminf_{\kappa\to-\infty} \nu(\kappa) = -\infty$, and $k(t)$ is generated via a nonlinear function of the product $\|e(t)\|\varphi(t)$. The feedback structure essentially exploits an intrinsic high-gain property of the system by ensuring that, if $(t, e(t))$ approaches the funnel boundary, then the gain attains values sufficiently large to preclude boundary contact.

1. Introduction

In the precursor (Ilchmann *et al*. 2002) to the present paper, the concept of a performance funnel was introduced in a context of tracking control for nonlinear systems. The basic problem addressed there was that of approximate tracking (with prescribed transient behaviour), by the system output y, of any absolutely continuous and bounded function r with essentially bounded derivative: the terminology "approximate tracking" means that, for any prescribed $\lambda > 0$, a control structure can be determined which ensures that the tracking error $e = y - r$ is ultimately bounded by λ (that is, $\|e(t)\| \le \lambda$ for all t sufficiently large); the terminology "with prescribed transient behaviour" means that, for some suitable prescribed function φ, the error function is required to satisfy $\|e(t)\| \le 1/\varphi(t)$ for all $t > 0$. The choice of φ determines the transient behaviour; moreover, by imposing the property $\liminf_{t\to\infty} \varphi(t) \ge 1/\lambda > 0$, the approximate tracking objective is assured. For example, with $\varphi\colon t \mapsto \min\{t/T, 1\}/\lambda$, the approximate tracking objective is achieved in prescribed time $T > 0$. Figure 1 encapsulates the approach: the function φ determines the performance funnel $\mathcal{F}_\varphi$, which may be identified with the graph of the set-valued map $t \mapsto \{v \mid \varphi(t)\|v\| < 1\}$. Simply stated, the control objective is to maintain the evolution of the tracking error in the funnel $\mathcal{F}_\varphi$. For reference signals of the generality considered in (Ilchmann *et al*. 2002) (namely, signals of class $W^{1,\infty}$), the function φ is required to be bounded and hence *exact* asymptotic tracking cannot be achieved. The purpose of the present note is to demonstrate that the boundedness condition on φ may be relaxed if one restricts the class of reference signals to coincide with the set of solutions of a fixed, stable, linear, homogeneous differential equation and confines attention to minimum-phase linear systems with sign-definite high-frequency gain. Under these restrictions, exact asymptotic tracking is achieved by

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International Journal of Control
ISSN 0020–7179 print/ISSN 1366–5820 online © 2006 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/00207170600708699

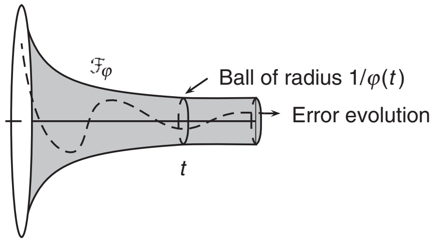

Figure 1. Performance funnel $\mathcal{F}_\varphi$

adopting an internal model (capable of replicating the reference signals) in conjunction with a performance funnel with radius asymptotic to zero and an output feedback structure akin to that in (Ilchmann *et al.* 2002 § 6.3). In an adaptive control context, the use of internal models in problems of asymptotic tracking for linear systems is well established (see, for example, Mårtensson 1986, Miller and Davison 1987, Helmke *et al.* 1990, Ilchmann 1993). We emphasize that the approach adopted in the present paper is non-adaptive: the control structure involves an internal model and a proportional feedback term, with gain determined by a measure of distance between the instantaneous tracking error and the funnel boundary; the latter feature is in contrast with the adaptive schemes where controller gains are dynamically generated via differential or integral equations.

2. Class of systems

We consider the class of m-input ($u(t) \in \mathbb{R}^m$), m-output ($y(t) \in \mathbb{R}^m$) linear systems of the form

$$\left.\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t), \quad x(0) = x^0 \in \mathbb{R}^n \\ y(t) &= C\,x(t), \end{aligned}\right\} \tag{2.1}$$

where the triple $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ has the following properties:

P1: *strict relative degree one with sign-definite high-frequency gain*, that is,

$$\langle x, CBx\rangle = 0 \iff x = 0,$$

P2: *minimum-phase*, that is,

$$\det\begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} \neq 0 \quad \text{for all } s \in \mathbb{C} \text{ with } \operatorname{Re} s \geq 0.$$

We remark that, in P1, it is not assumed that CB is symmetric and, under assumption P1, the minimum-phase property P2 is equivalent to the assumption that the system (2.1) has exponentially stable zero dynamics (this equivalence can also be deduced from Lemma 3.4).

2.1 *Control objectives, class of reference signals and performance funnel*

Let $\mathcal{M}$ denote the set of square real matrices having no eigenvalue with positive real part and such that every eigenvalue on the imaginary axis is semi-simple. The *reference signals* to be tracked are all functions $r\colon \mathbb{R}_+ \to \mathbb{R}^m$ the components r_i of which are solutions of the scalar differential equation $\alpha(\mathrm{d}/\mathrm{d}t)r_i(\cdot) = 0$, where $\alpha \in \mathbb{R}[s]$ is the characteristic polynomial of some $M \in \mathcal{M}$ (and so every such function r is bounded). We denote this reference signal class by

$$\mathcal{Y}_{\mathrm{ref}}(\alpha) := \left\{ r \in C^\infty(\mathbb{R}_+, \mathbb{R}^m) \;\middle|\; \begin{aligned} &\alpha\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) r(\cdot) = 0, \\ &\alpha(s) = \det[sI - M], \quad M \in \mathcal{M} \end{aligned} \right\}.$$

For example, the admissible reference signals are functions $t \mapsto r(t) \in \mathbb{R}^m$, the components of which are linear combinations of constants and sinusoids.

The first control objective is asymptotic (output) tracking of any reference signal $r \in \mathcal{Y}_{\mathrm{ref}}(\alpha)$. By this we mean a (dynamic) output feedback strategy which incorporates an internal model (capable of replicating the reference signal) and which ensures that $\lim_{t\to\infty}\big(y(t) - r(t)\big) = 0$ whilst maintaining boundedness of all the other signals. The second control objective is the prescribed transient behaviour of the error signal $e = y - r$. We capture both the objectives in the concept of a performance funnel

$$\mathcal{F}_\varphi := \left\{ (t, e) \in \mathbb{R}_+ \times \mathbb{R}^m \,\middle|\, \varphi(t)\,\|e\| < 1 \right\} \tag{2.2}$$

associated with a function φ (the reciprocal of which determines the funnel boundary) with the following properties

$$\left.\begin{aligned} &\text{(a)} \quad \varphi\colon \mathbb{R}_+ \to \mathbb{R}_+ \text{ is absolutely continuous and} \\ &\qquad\quad \text{non-decreasing;} \\ &\text{(b)} \quad \varphi(t) = 0 \iff t = 0; \\ &\qquad\quad \text{there exists } c > 1 \text{ such that:} \\ &\text{(c)} \quad \varphi(t) \leq c\,\varphi(t/2) \quad \text{for all } t \in \mathbb{R}_+; \\ &\text{(d)} \quad \dot{\varphi}(t) \leq c\,[1 + \varphi(t)] \quad \text{for almost all } t \in \mathbb{R}_+. \end{aligned}\right\} \tag{2.3}$$

For example, $t \mapsto \varphi(t) = t^a$, $a \geq 1$, satisfies (2.3) with $c = 2^a$. We record the following observation for later use.

Proposition 2.1: *Let φ be such that* (2.3) *holds. For every $p \geq \ln c/\ln 2$,*

$$0 < \varphi(t) \leq \varphi(1)\,[1 + ct^p] \quad \textit{for all } t > 0. \tag{2.4}$$

Proof: Since φ is non-decreasing with property (b), we have $0 < \varphi(t) \leq \varphi(1)$ for all $t \in (0, 1]$. Now, let $t \in (1, \infty)$ be arbitrary and choose $n \in \mathbb{N}$ such that $2^{n-1} \leq t \leq 2^n$ or, equivalently, $1/2 \leq t/2^n \leq 1$. Then, by (b), (c) and the non-decreasing property,

$$\begin{aligned}0 < \varphi(t) &\leq c\,\varphi(t/2) \leq \cdots \leq c^n\,\varphi(t/2^n) \leq c^n\,\varphi(1)\\ &= c\,\varphi(1)\,2^{(n-1)\ln c/\ln 2} \leq c\,\varphi(1)\,t^p.\end{aligned}$$

The claim (2.4) follows. □

Proposition 2.1 implies, in particular, that exponentially contracting funnels are excluded.

3. The control

Let $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ be such that P1 and P2 hold, and define

$$s(CB) := \begin{cases} +1, & \text{if } \langle x, CBx\rangle > 0\ \forall x \neq 0\\ -1, & \text{if } \langle x, CBx\rangle < 0\ \forall x \neq 0. \end{cases} \tag{3.5}$$

We will have occasions to consider the two possible cases: $s(CB)$ known or unknown *a priori* (the latter case is largely of academic interest).

3.1 *Internal model*

A body of work by Francis and Wonham in the 1970s (see, for example, Francis and Wonham 1975, Wonham 1976) led to the so-called Internal Model Principle, succinctly summarized in the context of linear systems in (Wonham 1979, p. 210) as "every good regulator must incorporate a model of the outside world". Recent extensions of this "principle" to a non-linear setting are contained in (Sontag 2003).

Let $\alpha \in \mathbb{R}[s]$ be the characteristic polynomial of some $M \in \mathcal{M}$ (and so every $r \in \mathcal{Y}_{\text{ref}}(\alpha)$ is bounded). Let $\beta \in \mathbb{R}[s]$ be a monic Hurwitz polynomial (i.e. all zeros of β lie in the open left-half complex plane) and such that α and β are the coprime of degree $p := \deg\beta = \deg\alpha$. Then

$$\lim_{s\to\infty} \beta(s)/\alpha(s) = 1. \tag{3.6}$$

The *internal model* is now defined to be the m-input, m-output linear system with transfer function

$$G_m(s) := \frac{\beta(s)}{\alpha(s)}\,I_m. \tag{3.7}$$

Let $(\hat{A}, \hat{b}, \hat{c}, 1) \in \mathbb{R}^{p\times p} \times \mathbb{R}^{p\times 1} \times \mathbb{R}^{1\times p} \times \mathbb{R}$ be a minimal state space realization of $\beta(s)/\alpha(s)$. Then a minimal state space realization of the internal model is given by

$$\left.\begin{aligned}\dot{\xi}(t) &= \widetilde{A}\,\xi(t) + \widetilde{B}\,w(t), \quad \xi(0) = \xi^0\\ u(t) &= \widetilde{C}\,\xi(t) + I_m\,w(t)\end{aligned}\right\} \tag{3.8}$$

with

$$\widetilde{A} = \operatorname{diag}\{\hat{A}, \ldots, \hat{A}\} \in \mathbb{R}^{mp\times mp}, \widetilde{B} = \operatorname{diag}\{\hat{b}, \ldots, \hat{b}\} \in \mathbb{R}^{mp\times m},$$
$$\widetilde{C} = \operatorname{diag}\{\hat{c}, \ldots, \hat{c}\} \in \mathbb{R}^{m\times mp}, \quad \xi^0 \in \mathbb{R}^{mp}.$$

We refer to $(\widetilde{A}, \widetilde{B}, \widetilde{C}, I_m)$ as the internal model (although, strictly speaking, the use of "the" here is incorrect as any quadruple in the similarity orbit of $(\widetilde{A}, \widetilde{B}, \widetilde{C}, I_m)$ also qualifies for the title "internal model").

3.2 *Feedback*

Let φ be such that (2.3) holds, and let $\mathcal{F}_\varphi$ be the associated performance funnel given by (2.2). Let $\nu\colon \mathbb{R} \to \mathbb{R}$ be any C^∞ function such that, for some strictly increasing, unbounded sequence (k_j) in $(1, \infty)$,

$$\nu(k_j)\,s(CB) \to \infty \quad \text{as } j \to \infty. \tag{3.9}$$

If $s(CB)$ is known *a priori*, then $\nu\colon k \mapsto k\,s(CB)$ suffices. If $s(CB)$ is unknown *a priori*, then any C^∞ function ν with the following properties suffices:

$$\limsup_{k\to\infty} \nu(k) = +\infty \quad \text{and} \quad \liminf_{k\to\infty} \nu(\mathrm{k}) = -\infty. \tag{3.10}$$

A simple example of a function satisfying (3.10) is $\nu\colon k \mapsto k\cos k$. In the latter case of unknown $s(CB)$, the role of the function ν is similar to the concept of a "Nussbaum" function in adaptive control. Note, however, that the requisite properties (3.10) are less restrictive than: (a) the "Nussbaum property"

$$\limsup_{k\to\infty} \frac{1}{k}\int_0^k \nu(\kappa)\,\mathrm{d}\kappa = \infty, \quad \liminf_{k\to\infty} \frac{1}{k}\int_0^k \nu(\kappa)\,\mathrm{d}\kappa = -\infty,$$

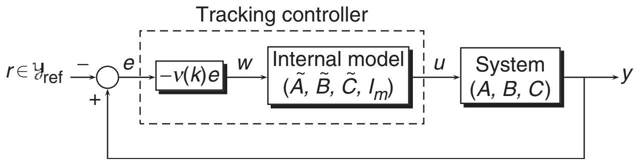

Figure 2. Tracking control with internal model.

as required in (Ye 1999), for example, or (b) the stronger "scaling invariant" Nussbaum property, as required in (Jiang *et al*. 2004), for example.

The control strategy is given by

$$\left.\begin{aligned} w(t) &= -\nu(k(t))\,[\,y(t) - r(t)],\\ k(t) &= \frac{1}{1 - \big(\varphi(t)\|y(t) - r(t)\|\big)^2} \end{aligned}\right\}, \tag{3.11}$$

in series with the internal model (3.8) (see figure 2).

3.3 *Closed-loop system*

For $r \in \mathcal{Y}_{\mathrm{ref}}$, let $\mathcal{D}_r \subset \mathbb{R}_+ \times \mathbb{R}^{n+mp}$ denote the connected, relatively open set

$$\mathcal{D}_r := \big\{(t,\zeta) \in \mathbb{R}_+ \times \mathbb{R}^{n+mp} |\ \varphi(t)\|\bar{C}\zeta - r(t)\| < 1\big\}. \tag{3.12}$$

The conjunction of (2.1), (3.8) and (3.11) yields the closed-loop initial-value problem (on $\mathcal{D}_r$)

$$\dot{\bar{x}}(t) = f(t, \bar{x}(t)), \quad \bar{x}^0 = \begin{bmatrix} x^0 \\ \xi^0 \end{bmatrix}, \tag{3.13}$$

where $f\colon \mathcal{D}_r \to \mathbb{R}^{n+mp}$ is given by

$$f(t,\zeta) := \bar{A}\zeta - \nu\big([1 - (\varphi(t)\|\bar{C}\zeta - r(t)\|)^2]^{-1}\big)\bar{B}[\bar{C}\zeta - r(t)], \tag{3.14}$$

with

$$\bar{A} := \begin{bmatrix} A & B\widetilde{C} \\ 0 & \widetilde{A} \end{bmatrix}, \quad \bar{B} := \begin{bmatrix} B \\ \widetilde{B} \end{bmatrix}, \quad \bar{C} := \begin{bmatrix} C, 0 \end{bmatrix}, \tag{3.15}$$

$$\bar{x}(t) := \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}.$$

By a *solution* of (3.13)–(3.15), we mean a continuously differentiable function $\bar{x}\colon [0,\omega) \to \mathbb{R}^{n+mp}$, with $0 < \omega \le \infty$ and $(t, \bar{x}(t)) \in \mathcal{D}_r$ for all $t \in [0,\omega)$, which satisfies (3.13) and $\bar{x}$ is said to be *the unique maximal solution* if the following holds

$$\tilde{x}\colon [0,\tilde{\omega}) \to \mathbb{R}^{n+mp} \text{ is a solution of (3.13)–(3.15)}$$
$$\Longrightarrow \tilde{\omega} \le \omega \quad \text{and} \quad \bar{x}|_{[0,\tilde{\omega})} = \tilde{x}.$$

Observe that f is locally Lipschitz on $\mathcal{D}_r$. The following is now a consequence of the standard theory of ordinary differential equations (see, for example (Walter 1998, Theorem IV, p. 108)).

Proposition 3.1: *Let $r \in \mathcal{Y}_{\mathrm{ref}}$ be arbitrary. For each $(x^0, \xi^0) \in \mathbb{R}^n \times \mathbb{R}^{mp}$, the initial value problem* (3.13)–(3.15) *has unique maximal solution $\bar{x}\colon [0,\omega) \to \mathbb{R}^{n+mp}$. Moreover, if $\omega < \infty$, then, for every compact $\mathcal{C} \subset \mathcal{D}_r$, there exists $t \in [0,\omega)$ such that $(t, \bar{x}(t)) \notin \mathcal{C}$.*

3.4 *Main result*

Theorem 3.2: *Let $(A,B,C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ have strict relative degree one, sign-definite high-frequency gain, and be minimum-phase. Let φ satisfy* (2.3), *let $\mathcal{F}_\varphi$ be the performance funnel* (2.2) *associated with φ, and let $r \in \mathcal{Y}_{\mathrm{ref}}(\alpha)$. Then the feedback* (3.11) *applied in series with the internal model* (3.8) *yields the initial-value problem* (3.13)–(3.15) *which, for every $(x^0, \xi^0) \in \mathbb{R}^n \times \mathbb{R}^{mp}$, has unique maximal solution $\bar{x}\colon \mathbb{R}_+ \to \mathbb{R}^{n+mp}$. Moreover,*

(i) *the functions $\bar{x}$, $y = \bar{C}\bar{x}$, and*

$$k\colon t \mapsto \big[1 - (\varphi(t)\|y(t) - r(t)\|)^2\big]^{-1},$$
$$u\colon t \mapsto -\nu(k(t))[y(t) - r(t)]$$

are bounded;

(ii) *there exists $\varepsilon \in (0,1)$ such that, for all $t \ge 0$, $\varphi(t)\,\|y(t) - r(t)\| \le 1 - \varepsilon$;*

(iii) *if φ is unbounded, then $(y(t) - r(t), u(t)) \to (0,0)$ as $t \to \infty$.*

Remark 3.3: In the specific case of positive-definite CB and zero reference signal $r \equiv 0$, it is shown in (Ilchmann *et al*. 2002) that the assertions of Theorem 3.2 hold for the feedback $u = -ke$ without recourse to an internal model.

The proof of Theorem 3.2 invokes three lemmas; we briefly digress to present these.

3.5 *Three technical lemmas*

The first lemma is well known and is a re-statement of (Ilchmann 1993, Lemma 2.1.3).

Lemma 3.4: *Assume that $(A,B,C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ has strict relative degree one. Let $V \in \mathbb{R}^{n\times(n-m)}$ be* *such that* $\operatorname{im} V = \ker C$ (*of dimension* $n-m$) *and write*

$$N := (V^T V)^{-1} V^T \big[I_n - B(CB)^{-1} C \big].$$

Then

$$L = \begin{bmatrix} C \\ N \end{bmatrix}$$

is invertible, with inverse $L^{-1} = \big[B(CB)^{-1}, V \big]$ *and*

$$LAL^{-1} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad LB = \begin{bmatrix} CB \\ 0 \end{bmatrix}, \quad CL^{-1} = [I_m \ \ 0]$$

where $A_1 \in \mathbb{R}^{m\times m}$ (*with* A_2, A_3, A_4 *of conforming formats*). *Furthermore*, (A,B,C) *is minimum phase if, and only if*, A_4 *is Hurwitz*.

Lemma 3.5: *Let* $(A,B,C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ *be minimum phase with strict relative degree one and sign-definite high-frequency gain. If* $(\widetilde{A}, \widetilde{B}, \widetilde{C}, I_m)$ *is a minimal realization of the internal model as specified in subsection* 3.1, *then* $(\bar{A}, \bar{B}, \bar{C})$, *as defined in* (3.15), *is minimum phase with strict relative degree one and sign-definite high-frequency gain.*

Proof: Clearly, $\bar{C}\bar{B} = CB$ and so the system $(\bar{A}, \bar{B}, \bar{C})$ has strict relative degree one and sign-definite high-frequency gain.

It remains to show that

$$\det \begin{bmatrix} sI - \bar{A} & \bar{B} \\ \bar{C} & 0 \end{bmatrix} \neq 0 \quad \text{for all } s \in \mathbb{C} \text{ with } \operatorname{Re} s \geq 0.$$

Since $(\hat{A}, \hat{b})$ is a controllable pair, the Hautus condition implies that $[sI - \hat{A}, \hat{b}]$ has full rank p for all $s \in \mathbb{C}$, whence

$$\operatorname{rank} \big[sI - \widetilde{A} \quad \widetilde{B} \big] = mp \quad \text{for all } s \in \mathbb{C}.$$

By the minimum-phase property of (A,B,C), we have

$$\operatorname{rank} \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} = n + m \quad \text{for all } s \in \mathbb{C} \text{ with } \operatorname{Re}(s) \geq 0,$$

and so

$$\operatorname{rank} \begin{bmatrix} sI - \bar{A} & \bar{B} \\ \bar{C} & 0 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} sI - A & -B\widetilde{C} & B \\ 0 & sI - \widetilde{A} & \widetilde{B} \\ C & 0 & 0 \end{bmatrix} = n + mp + m$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s \geq 0$, and the claim follows. □

A proof of the following lemma can be found in (Miller and Davison 1991), see also (Ilchmann 1993, Lemma 5.1.2).

Lemma 3.6: *Let* $(A,B,C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ *be minimum phase with strict relative degree one and sign-definite high-frequency gain. If* $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ *is a minimal realization of the internal model as specified in subsection* 3.1, *then, for any* $r \in \mathcal{Y}_{\text{ref}}(\alpha)$, *there exists* $\rho^0 \in \mathbb{R}^{n+mp}$ *such that*

$$\left.\begin{aligned} \dot{\rho}(t) &= \bar{A}\,\rho(t), \quad \rho(0) = \rho^0 \\ r(t) &= \bar{C}\,\rho(t), \end{aligned}\right\} \tag{3.16}$$

where $\bar{A}$ and $\bar{C}$ are given by (3.15).

3.6 *Proof of Theorem 3.2*

By Proposition 3.1, (3.13)–(3.15) has unique maximal solution $\bar{x}$: $[0,\omega) \to \mathbb{R}^{n+mp}$, with $0 < \omega \leq \infty$.

By Lemma 3.6, there exists $\rho^0 \in \mathbb{R}^{n+mp}$ such that $r(\cdot) = \bar{C}\rho(\cdot)$, where ρ: $t \mapsto \big(\exp \bar{A}t\big)\rho^0$. Writing

$$x_e(t) = \bar{x}(t) - \rho(t), \quad e(t) = y(t) - r(t),$$

together with (3.13)–(3.15) gives,

$$\left.\begin{aligned} \dot{x}_e(t) &= \bar{A}\,x_e(t) - \nu(k(t))\bar{B}\,e(t), \quad x_e(0) = x_e^0 := \bar{x}^0 - \rho^0, \\ e(t) &= \bar{C}\,x_e(t), \\ k(t) &= \Big[1 - \big(\varphi(t)\|e(t)\|\big)^2\Big]^{-1} \\ &\forall t \in [0,\omega). \end{aligned}\right\} \tag{3.17}$$

By Lemma 3.5, $(\bar{A}, \bar{B}, \bar{C})$ is minimum phase with strict relative degree one, and so, by Lemma 3.4, there exists N such that

$$L := \begin{bmatrix} \bar{C} \\ N \end{bmatrix}$$

is invertible and the transformation

$$\begin{bmatrix} \bar{C} \\ N \end{bmatrix} x_e(t) = \begin{bmatrix} e(t) \\ z(t) \end{bmatrix}$$

converts (3.17) into the equivalent form

$$\left.\begin{aligned} \dot{e}(t) &= A_1\,e(t) + A_2\,z(t) - \nu(k(t))\,CB\,e(t) \\ \dot{z}(t) &= A_3\,e(t) + A_4\,z(t) \\ k(t) &= \Big[1 - \big(\varphi(t)\|e(t)\|\big)^2\Big]^{-1} \end{aligned}\right\} \quad \forall t \in [0,\omega), \tag{3.18}$$

where $A_4 \in \mathbb{R}^{(n+m(p-1))\times(n+m(p-1))}$ is Hurwitz and we have invoked the equality $\bar{C}\bar{B} = CB$. Since $(t, \bar{x}(t)) \in \mathcal{D}_r$ for all $t \in [0, \omega)$, we have

$$\varphi(t)\|e(t)\| < 1 \quad \forall t \in [0, \omega) \tag{3.19}$$

and so e is bounded, which, together with the Hurwitz property of A_4 and the second of equations (3.18), implies that z is bounded. It immediately follows that x_e is bounded, whence boundedness of $\bar{x} = x_e + \rho$.

Writing $e^0 = \bar{C}x_e^0$, $z^0 = Nx_e^0$ and defining

$$\begin{aligned} q_0(t) &:= A_2 \exp(A_4 t)z^0, \\ q_1(t) &:= A_1 e(t) + A_2 \int_0^t \exp(A_4(t-s))A_3 e(s)\mathrm{d}s, \quad \forall t \in [0, \omega), \end{aligned} \tag{3.20}$$

then the first two equations in (3.18) are equivalent to

$$\dot{e}(t) = q_0(t) + q_1(t) - \nu(k(t))\, CB\, e(t) \quad \forall t \in [0, \omega). \tag{3.21}$$

Since A_4 is Hurwitz, there exist $c_1, \mu > 0$ such that

$$\|q_0(t)\| = \|A_2 \exp(A_4 t)z^0\| \le c_1\, e^{-\mu t} \quad \forall t \in [0, \omega) \tag{3.22}$$

and

$$\begin{aligned} \|q_1(t)\| &\le \|A_1\|\|e(t)\| + c_1 \left(\int_0^{t/2} + \int_{t/2}^t\right) e^{-\mu(t-s)}\|e(s)\|\mathrm{d}s \\ &\le \|A_1\|\|e(t)\| + \frac{c_1}{\mu}\left[e^{-\mu\, t/2} \max_{s\in[0,\, t/2]} \|e(s)\| \right. \\ &\quad \left. + \max_{s\in[t/2,\, t]} \|e(s)\|\right] \quad \forall t \in [0, \omega). \end{aligned} \tag{3.23}$$

By boundedness of e, together with (3.19) and invoking property (2.3d) of φ, we may infer the existence of $c_2 > 0$ such that

$$\begin{aligned} \varphi(t)\, \dot{\varphi}(t)\, \|e(t)\|^2 &\le c[1 + \varphi(t)]\, \varphi(t)\, \|e(t)\|^2 \\ &\le c\,[1 + 2\varphi^2(t)]\, \|e(t)\|^2 \le c\,[\|e(t)\|^2 + 2] \\ &\le c_2 \quad \text{for almost all } t \in [0, \omega). \end{aligned} \tag{3.24}$$

Since CB is sign definite, there exists $c_3 > 0$ such that

$$\frac{1}{2} c_3\, \|e\|^2 \le |\langle e, CB\, e\rangle| \quad \forall\, e \in \mathbb{R}^m. \tag{3.25}$$

Now we are in a position to prove the boundedness of k. Define $\tilde{\nu}\colon \mathbb{R} \to \mathbb{R}$ as follows

$$\tilde{\nu}(k) := \nu(k)\, s(CB).$$

By property (3.9) of ν, there exists a strictly increasing unbounded sequence (k_j) in $(1, \infty)$ such that $\tilde{\nu}(k_j) \to \infty$ as $j \to \infty$. Passing to a subsequence if necessary, we may assume that the sequence $(\tilde{\nu}(k_j))$ is in $(0, \infty)$ and is strictly increasing. Seeking a contradiction, suppose that k is unbounded. For each $j \in \mathbb{N}$, define

$$\begin{aligned} \tau_j &:= \inf\{t \in [0, \omega) | k(t) = k_{j+1}\} \\ \sigma_j &:= \sup\{t \in [0, \tau_j] | \tilde{\nu}(k(t)) = \tilde{\nu}(k_j)\} \\ \tilde{\sigma}_j &:= \sup\{t \in [0, \tau_j] | k(t) = k_j\} \le \sigma_j. \end{aligned}$$

Observe that

$$k(\tau_j) > k(\sigma_j) \quad \forall j \in \mathbb{N}. \tag{3.26}$$

Furthermore, for all $j \in \mathbb{N}$ and all $t \in [\sigma_j, \tau_j]$, we have $k(t) \ge k_j$ and $\tilde{\nu}(k(t)) \ge \tilde{\nu}(k_j)$. Therefore,

$$\begin{aligned} &1 > (\varphi(t)\|e(t)\|)^2 \ge 1 - \frac{1}{k_j} \ge 1 - \frac{1}{k_1} =: c_4 > 0 \\ &\forall t \in [\sigma_j, \tau_j] \quad \forall j \in \mathbb{N}, \end{aligned} \tag{3.27}$$

and since φ is non-decreasing, we arrive at

$$\begin{aligned} &\max_{s\in[t/2,\, t]} \|e(s)\| < \frac{1}{\varphi(t/2)} \le \frac{\varphi(t)}{\sqrt{c_4}\, \varphi(t/2)}\, \|e(t)\| \\ &\forall t \in [\sigma_j, \tau_j] \quad \forall j \in \mathbb{N}. \end{aligned} \tag{3.28}$$

By (3.23) and (3.28), together with boundedness of e and property (2.3c) of φ, we may infer the existence of $c_5 > 0$ such that

$$\|q_1(t)\| \le c_5\left[e^{-\mu t/2} + \|e(t)\|\right] \quad \forall t \in [\sigma_j, \tau_j] \quad \forall j \in \mathbb{N}. \tag{3.29}$$

Invoking (3.24), (3.22), (3.25), (3.27), recalling that $\varphi(t)\|e(t)\| < 1$ for all $t \in [0, \omega)$, and noting that, by Proposition 2.1, the functions $t \mapsto \varphi(t)e^{-\mu t}$ and $t \mapsto \varphi(t)e^{-\mu t/2}$ are bounded, we may conclude the existence of $c_6 > 0$ such that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} k(t) = &\; k^2(t)\Big[2\, \varphi(t)\dot{\varphi}(t)\, \|e(t)\|^2 + 2\, \varphi^2(t)\, \langle e(t), q_0(t) \\ &+ q_1(t) - \nu(k(t))\, CB\, e(t)\rangle\Big] \\ \le &\; k^2(t)\Big[2\, c_2 + 2\, \varphi(t)\big[\|q_0(t)\| + \|q_1(t)\|\big] \\ &- 2\, \varphi^2(t)\, \tilde{\nu}(k(t))\, |\langle e(t),\, CB\, e(t)\rangle|\,\Big] \\ \le &\; k^2(t)\Big[2\, c_2 + 2\, c_1 \varphi(t)\, e^{-\mu t} + 2\, c_5\, \varphi(t) \\ &\times \left[e^{-\mu t/2} + \|e(t)\|\right] - c_3\, \varphi^2(t)\, \tilde{\nu}(k(t))\, \|e(t)\|^2\Big] \\ \le &\; k^2(t)[c_6 - c_3\, c_4\, \tilde{\nu}(k(t))] \\ &\text{for almost all } t \in [\sigma_j, \tau_j] \text{ and all } j \in \mathbb{N}. \end{aligned}$$

Let $j^* \in \mathbb{N}$ be sufficiently large to that $c_6 - c_3 c_4 \tilde{\nu}(k_{j^*}) < 0$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t} k(t) < 0 \quad \text{for almost all} \quad t \in [\sigma_{j^*}, \tau_{j^*}],$$

which contradicts (3.26). This proves the boundedness of k.

Next, we show the boundedness of u. Since k is bounded, there exists $\varepsilon > 0$ such that $\varphi(t)\|e(t)\| \le 1 - \varepsilon$ for all $t \in [0, \omega)$. By boundedness of e, z and k, it follows that u is bounded.

We proceed to prove that $\omega = \infty$. Suppose that ω is finite. Let $c_7 > 0$ be such that $\|x_e(t)\| \le c_7$ for all $t \in [0, \omega)$, and set

$$\mathcal{C} := \left\{ (t, \zeta) \in \mathcal{D}_r \,\middle|\, \begin{array}{l} \varphi(t)\,\|\bar{C}\zeta - r(t)\| \le 1 - \varepsilon, \\ \|\zeta\| \le c_7, t \in [0, \omega] \end{array} \right\}.$$

Then $\mathcal{C}$ is a compact subset of $\mathcal{D}_r$ with the property that $(t, \bar{x}(t)) \in \mathcal{C}$ for all $t \in [0, \omega)$. This contradicts Proposition 3.1. Therefore, the supposition that ω is finite is false. This completes the proof of assertions (i)–(iii).

It remains only to establish the assertion (iv). Assume that φ is unbounded. Then $\|e(t)\| < 1/\varphi(t) \to 0$ as $t \to \infty$. By boundedness of k, we have $u(t) = -\nu(k(t))e(t) \to 0$ as $t \to \infty$. □

4. Example

Let (A, b, c) be a single-input, single-output minimum-phase system with positive high-frequency gain $cb > 0$. Assume that the class of reference signals $r: \mathbb{R}_+ \to \mathbb{R}$ comprises all the linear combinations of constant functions and the sinusoidal functions of period 2π. Choosing as internal model the linear system with transfer function

$$\frac{\beta(s)}{\alpha(s)} = \frac{(s+1)^3}{s(s^2+1)},$$

and selecting the funnel function $t \mapsto \varphi(t) := t^2$, then the feedback

$$u(t) = -k(t)e(t), \quad k(t) = \frac{1}{1 - (t^2 e(t))^2}, \quad e(t) = y(t) - r(t),$$

in series with the internal model, ensures the asymptotic tracking of every admissible reference signal r and achieves a tracking error decay rate of the order t^{-2}. In the specific case

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad c = [1\ \ 0\ \ 0],$$

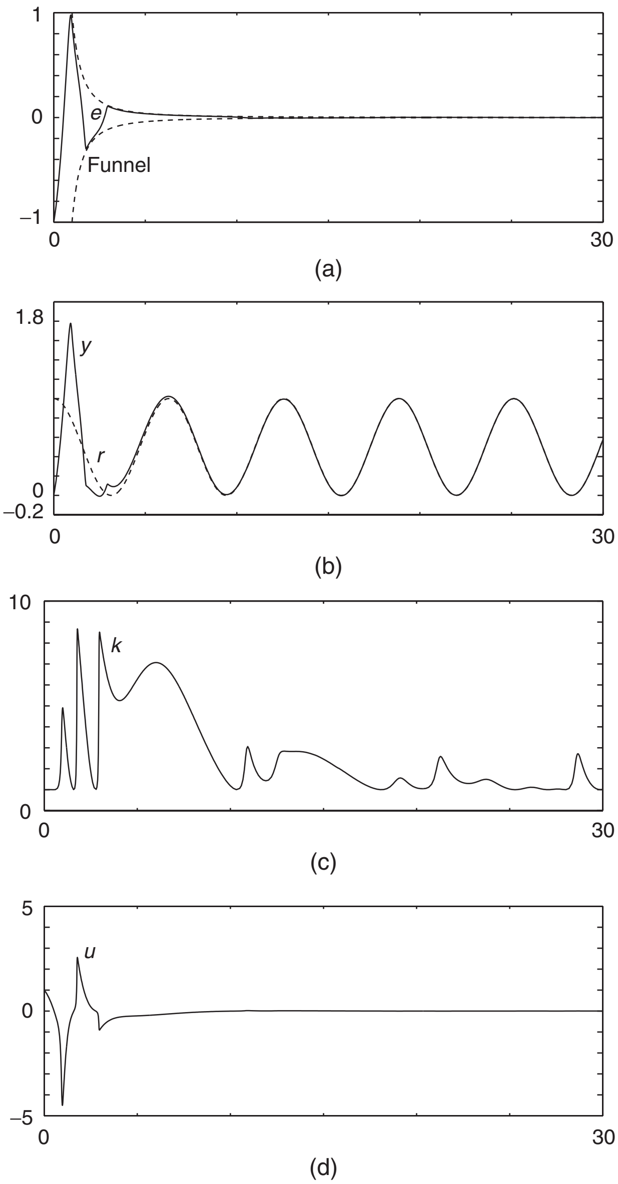

Figure 3. Example (a) The funnel and tracking error e, (b) The reference r and output y, (c) The gain function k, (d) The control u.

with zero initial conditions and reference signal

$$r: t \mapsto \frac{1}{2}[1 + \cos t],$$

the behaviour of the feedback system is depicted in figure 3(a–d).

5. Conclusion

We have presented a "funnel" controller for m-input, m-output, linear, minimum-phase systems which have strict relative degree one. This controller achieves asymptotic tracking – with prescribed transient behaviour – of signals $r: \mathbb{R}_+ \to \mathbb{R}^m$, the components of which are solutions of a scalar ordinary differential equation. The novelty – compared to the previous contribution on funnel control in (Ilchmann *et al.* 2002) – is that the asymptotic tracking is *exact* whereas, in (Ilchmann *et al.* 2002) only *approximate* tracking is achieved. Otherwise stated, the funnels in the present paper are permitted to have radius $1/\varphi(t)$ converging to zero as $t \to \infty$ whereas, in (Ilchmann *et al.* 2002), boundedness of the function φ is required. However, the enhanced tracking performance of the present paper is achieved at the expenditure of a reference signal class which is more restrictive than that considered in (Ilchmann *et al.* 2002). This restriction underpins a linear internal model approach to control design in the present paper, an approach which differs fundamentally from that adopted in (Ilchmann *et al.* 2002). A notable feature of the funnel control is the non-dynamic nature of the feedback gain: this contrasts favourably with the existing adaptive designs for stabilizing or tracking control of the linear systems (see, e.g. Byrnes and Willems 1984, Helmke *et al.* 1990, Ilchmann 1993, Mårtensson 1986, Miller and Davison 1987, 1991) where the feedback gain is dynamically generated. It remains to investigate how far the present approach carries over to certain classes of nonlinear systems: in this context, the recent results on the use of *nonlinear* internal models in regulator design (Byrnes and Isidori 2004) may be of relevance.

Acknowledgement

Based on work supported in part by the UK Engineering & Physical Sciences Research Council (GR/S94582/01).

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