Low-gain integral control of well-posed systems subject to input hysteresis: an input-output approach

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Introduction

We consider a general class of hysteresis operators with certain natural monotonicity and Lipschitz continuity properties (recall that a hysteresis operator is defined to be a causal and rate-independent operator mapping the space of continuous functions into itself). The class of hysteresis nonlinearities under consideration contains, in particular, backlash, elastic plastic and operators of Prandtl and Preisach type. It is shown that closing the loop around an L^2-stable, time-invariant linear system, subject to input hysteresis of this class and compensated by an integral controller, guarantees tracking of constant reference signals, provided that (a) the steady-state gain of the linear part of the plant is positive, (b) the positive time-dependent integrator gain is ultimately smaller than a certain constant given by a positive-real condition on the linear system and (c) the reference value is feasible in a natural sense. Our input-output approach complements and extends earlier work in [2] where a state-space approach to low-gain integral control of exponentially stable regular infinite-dimensional systems with input hysteresis is developed.

A class of hysteresis operators

A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *time transformation* if f is continuous and non-decreasing with $f(0) = 0$ and $\lim_{t\to\infty} f(t) = \infty$; in other words f is a time transformation if it is continuous, non-decreasing and surjective. An operator $\Phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ is called *rate independent* if, for every time transformation f,

$$(\Phi(u \circ f))(t) = (\Phi(u))(f(t)), \quad \forall\, u \in C(\mathbb{R}_+), \quad \forall\, t \in \mathbb{R}_+ .$$

We say that $\Phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ is a *hysteresis operator* if Φ is causal and rate independent. The *numerical value set* NVS Φ of a hysteresis operator Φ is defined by

$$\mathrm{NVS}\,\Phi := \{(\Phi(u))(t) : u \in C(\mathbb{R}_+),\, t \in \mathbb{R}_+\} .$$

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A function $u \in C(\mathbb{R}_+)$ is called *ultimately non-decreasing* if there exists $\tau \in \mathbb{R}_+$ such that u is non-decreasing on $[\tau, \infty)$; u is said to be *approximately ultimately non-decreasing*, if for all $\varepsilon > 0$, there exists an ultimately non-decreasing function $v \in C(\mathbb{R}_+)$ such that

$$|u(t) - v(t)| \le \varepsilon\,, \quad \forall\, t \in \mathbb{R}_+\,.$$

For $w \in C([0,\alpha])$ (with $\alpha \ge 0$) and $\gamma, \delta > 0$, we define

$$\mathcal{C}(w;\delta,\gamma) := \{v \in C([0,\alpha+\gamma]) : v|_{[0,\alpha]} = w, \quad \sup_{t\in[\alpha,\alpha+\gamma]} |v(t) - w(\alpha)| \le \delta\}\,.$$

We impose the following six conditions on the hysteresis operator $\Phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$:

(N1) $\Phi(W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)) \subset W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$;

(N2) Φ is monotone in the sense that, for all $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$,

$$(\Phi(u))'(t)u'(t) \ge 0\,, \quad \text{a.e. } t \in \mathbb{R}_+\,;$$

(N3) there exists $\lambda > 0$ such that for all $\alpha \ge 0$ and $w \in C([0,\alpha])$, there exist numbers $\gamma, \delta > 0$ such that

$$\sup_{t\in[\alpha,\alpha+\gamma]} |(\Phi(u))(t) - (\Phi(v))(t)| \le \lambda \sup_{t\in[\alpha,\alpha+\gamma]} |u(t) - v(t)|\,, \quad \forall\, u, v \in \mathcal{C}(w;\delta,\gamma)\,;$$

(N4) for all $a > 0$ and all $u \in C([0,a),\mathbb{R})$, there exist $\gamma_1, \gamma_2 > 0$ such that

$$\sup_{t\in[0,\tau]} |(\Phi(u))(t)| \le \gamma_1 + \gamma_2 \sup_{t\in[0,\tau]} |u(t)|\,, \quad \forall\, \tau \in [0,a)\,;$$

(N5) if $u \in C(\mathbb{R}_+)$ is approximately ultimately non-decreasing and $\lim_{t\to\infty} u(t) = \infty$, then $\Phi(u)(t)$ and $\Phi(-u)(t)$ converge to $\sup \mathrm{NVS}\,\Phi$ and $\inf \mathrm{NVS}\,\Phi$, respectively, as $t \to \infty$;

(N6) if, for $u \in C(\mathbb{R}_+)$, $\lim_{t\to\infty}(\Phi(u))(t) \in \mathrm{int}\,\mathrm{NVS}\,\Phi$, then u is bounded.

It is not difficult to see that (N5) implies that $\mathrm{NVS}\,\Phi$ is an interval. The set of all hysteresis operators satisfying (N1)-(N6) is denoted by $\mathcal{N}(\lambda)$, where $\lambda > 0$ is the constant associated with (N3). It is well-known that many standard hysteresis nonlinearities which are important in control engineering are contained in $\mathcal{N}(\lambda)$ for some suitable $\lambda > 0$: this applies in particular to backlash (or play), plastic-elastic (or stop) and large classes of Prandtl and Preisach operators (see [2, 3]).

Low-gain integral control in the presence of hysteresis

Consider the feedback system shown in Figure 1, where $\rho \in \mathbb{R}$ is a constant, $\kappa\colon \mathbb{R}_+ \to \mathbb{R}$ is a time-varying gain, the operator $G : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is linear, Φ is a hysteresis operator and the function g models the effect of non-zero initial conditions of the system with input-output operator G.

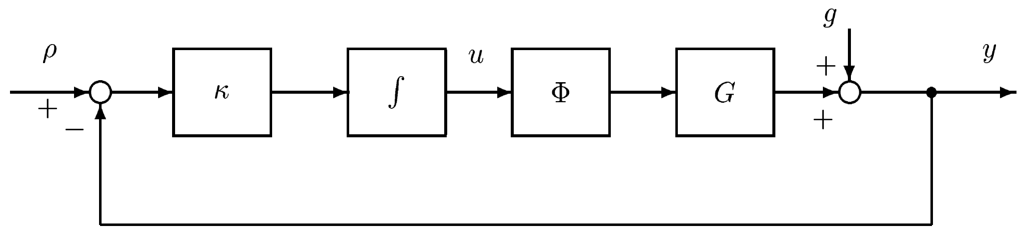

Figure 1

Since shift-invariance implies causality, G can be extended to a shift-invariant operator mapping $L^2_{\mathrm{loc}}(\mathbb{R}_+)$ into itself. We will not distinguish notationally between G and its extension. Denoting the transfer function of G by $\mathbf{G}$, respectively, we have that $\mathbf{G}$ is holomorphic and bounded in the open right-half plane. We assume that

(L) The limit $\mathbf{G}(0) := \lim_{s\to 0,\,\mathrm{Re}\,s>0} \mathbf{G}(s)$ exists, $\mathbf{G}(0) > 0$ and

$$\limsup_{s\to 0,\,\mathrm{Re}\,s>0} |(\mathbf{G}(s) - \mathbf{G}(0))/s| < \infty\,.$$

The feedback system shown in Figure 1 is described by the following abstract Volterra integro-differential equation

$$u' = \kappa(\rho - g - G\Phi(u))\,, \quad u(0) = u^0 \in \mathbb{R}\,, \tag{1}$$

The objective in this subsection is to determine gain functions κ such that the tracking error

$$e(t) := \rho - y(t) = \rho - g(t) - (G\Phi(u))(t)$$

becomes small in a certain sense as $t \to \infty$. For example, we might want to achieve "tracking in measure", i.e., for every $\varepsilon > 0$, the Lebesgue measure of the set $\{\tau \geq t : |e(\tau)| \geq \varepsilon\}$ tends to 0 as $t \to \infty$, or the aim might be "asymptotic tracking", that is $\lim_{t\to\infty} e(t) = 0$. Trivially, tracking in measure is guaranteed if $e \in L^p(\mathbb{R}_+)$ for some $p \in (0, \infty)$.

Setting

$$f(G) := \operatorname*{ess\,inf}_{\omega\in\mathbb{R}} \mathrm{Re}(\mathbf{G}(i\omega)/i\omega)\,,$$

It follows from assumption (L) that $-\infty < f(G) \leq 0$. Hence, for given $\varepsilon \in (0, 1)$, the positive-real condition

$$1 + \delta \mathrm{Re}(\mathbf{G}(i\omega)/i\omega) \geq \varepsilon\,, \quad \text{for a.e. } \omega \in \mathbb{R}\,,$$

holds, provided that $\delta > 0$ is sufficiently small.

We are now in the position to state the main result of this paper.

Theorem 1 *Let $G : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ be a linear bounded shift-invariant operator with transfer function $\mathbf{G}$. Assume that assumption* (L) *holds, $g \in L^2(\mathbb{R}_+)$, $\Phi \in \mathcal{N}(\lambda)$, $\rho \in \mathbb{R}$ is such that $\rho/\mathbf{G}(0) \in \overline{\mathrm{NVS}}\,\Phi$ and $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ is measurable and bounded with*

$$\limsup_{t\to\infty} \kappa(t) < 1/|\lambda f(G)|\,,$$

where $1/0 := \infty$. Then there exists a unique solution $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ of (1) *and the following statements hold.*

(1) $(\Phi(u))' \in L^2(\mathbb{R}_+)$ *and the limit* $l := \lim_{t\to\infty}(\Phi(u))(t)$ *exists and is finite.*

(2) *The signal* $y = g + G\Phi(u)$ *(see* Figure 1*) can be split into* $y = y_1 + y_2$*, where* y_1 *is continuous and satisfies*

$$\lim_{t\to\infty} y_1(t) = \mathbf{G}(0)l,$$

and $y_2 \in L^2(\mathbb{R}_+)$*. Under the additional assumptions that* $\lim_{t\to\infty} g(t) = 0$ *and the impulse response of* G *is a finite Borel measure, we have*

$$\lim_{t\to\infty} y_2(t) = 0\,.$$

(3) *If* $\kappa \notin L^1(\mathbb{R}_+)$*, then* $\lim_{t\to\infty} y_1(t) = \rho$ *and the error signal* e *can be split into* $e = e_1 + e_2$*, where* e_1 *is continuous with* $\lim_{t\to\infty} e_1(t) = 0$ *and* $e_2 \in L^2(\mathbb{R}_+, \mathbb{R})$*. Under the additional assumptions that* $\lim_{t\to\infty} g(t) = 0$ *and the impulse response of* G *is a finite Borel measure, we have*

$$\lim_{t\to\infty} e(t) = 0\,.$$

(4) *If* $\rho/\mathbf{G}(0)$ *is an interior point of* NVS Φ*, then* u *is bounded.*

Remark (a) Statement (3) of Theorem 1 implies tracking in measure. Under the assumption that ess $\lim_{t\to\infty} g(t) = 0$ and the impulse response of G is a finite Borel measure, statement (3) of Theorem 1 guarantees asymptotic tracking.

(b) Note that it is not necessary to know $f(G)$ or the constant λ in order to apply Theorem 1. If κ is chosen such that $\kappa(t) \to 0$ and $\kappa \notin L^1(\mathbb{R}_+)$ (e.g., $\kappa(t) = (1+t)^{-p}$ with $p \in (0,1)$), then the conclusions of statement (3) hold. However, from a practical point of view, gain functions κ with $\lim_{t\to\infty} \kappa(t) = 0$ might not be appropriate, since the system essentially operates in open loop as $t \to \infty$. In [4] it has been shown how $|f(G)|$ (or upper bounds for $|f(G)|$) can be obtained from frequency-response experiments performed on the linear part of the plant.

(c) The proof of Theorem 1 is based on a result of circle-criterion type recently published in [1]. Theorem 1 can be applied to strongly stable well-posed infinite-dimensional state-space systems. In particular, Theorem 1 can be viewed as an extension of the main result in [2] which applies to exponentially stable regular systems compensated by an integral controller with sufficiently small constant gain.

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