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Controlled functional differential equations and adaptive tracking[☆]

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Abstract

Adaptive tracking control of a class $\mathscr{N}$ of single-input, single-output systems described by nonlinear functional differential equations is considered: the control objective is that of tracking, by the system output, of reference signals of class $\mathscr{R}$ (absolutely continuous and bounded with essentially bounded derivative). A $(\mathscr{N},\mathscr{R})$-universal servomechanism, in the form of an adaptive error feedback strategy incorporating gains of Nussbaum type, is developed which, for every system of class $\mathscr{N}$ and every reference signal of class $\mathscr{R}$, ensures either (i) practical tracking (in the sense that prespecified asymptotic tracking accuracy, quantified by $\lambda > 0$, is assured), or (ii) asymptotic tracking (in the sense that the tracking error approaches zero). The first case (i) is achievable by *continuous* feedback; the second case (ii) necessitates *discontinuous* feedback. Both cases are developed within a framework of functional differential inclusions.

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1. Introduction

In [11], a universal stabilizer (adaptive output feedback control) was developed for a class of systems, modelled by nonlinear functional differential equations, having structure represented by Fig. 1 in the particular case wherein $r = 0 = v$.

The dynamic block Σ_1 (a forced nonlinear ordinary differential equation of the form $\dot{y} = g(p, y) + w + bu$, where b is a nonzero constant and $p \in L^\infty(\mathbb{R})$ is a bounded perturbation term), which can be influenced directly by the control u, is also driven by the output w from the dynamic block Σ_2. Viewed abstractly, the block Σ_2 can be regarded as a causal operator $\hat{T}$ which maps y to w.

In the present paper, we revisit systems of the above form, viz. $\dot{y} = g(p, y) + \hat{T}y + bu$, but now in the context of a tracking problem wherein $r \neq 0$ represents a reference signal and v represents an additive measurement disturbance signal on the system output y. Under stronger hypotheses (vis à vis those posited in the case of the stabilization problem ($r = 0 = v$) considered in [11]) on the operator $\hat{T}$ representing the subsystem Σ_2, we construct an adaptive error feedback strategy (parameterized by $\lambda \geqslant 0$) which ensures that, in the

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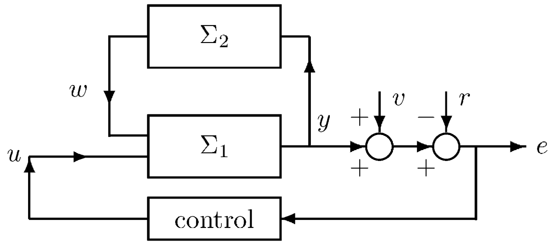

Fig. 1. The generic system.

absence of output measurement disturbance ($v=0$) and for every system of the admissible class $\mathcal{N}$ and every reference signal $r \in \mathcal{R} := W^{1,\infty}(\mathbb{R})$ (the space of bounded absolutely continuous functions with essentially bounded derivatives), the closed-loop system is such that all signals remain bounded and the tracking error approaches the set $\{e \,|\, |e| \leqslant \lambda\}$ as $t \to \infty$. Note that, in the case $\lambda = 0$, exact asymptotic tracking of the reference signal by the output is assured, that is, $y(t) \to r(t)$ as $t \to \infty$; however, this performance is achieved at the cost of discontinuous error feedback. In the case $\lambda > 0$, the error feedback is continuous; however, the price paid for this feedback regularity is that exact asymptotic tracking cannot be guaranteed—instead, approximate tracking is achieved with asymptotic error quantified by $\lambda > 0$. At first glance, these results seem to contradict the *internal model principle* which, loosely speaking, states that, if a smooth system Σ can achieve exact asymptotic tracking of reference signals of a prescribed class $\mathcal{R}$, then Σ must necessarily contain a subsystem Σ_s which can itself generate all signals of class $\mathcal{R}$ (see, for example, the seminal work [4] in the case of linear systems and the recent work [12] in the case of nonlinear systems). However, this apparent paradox is resolved on noting that, in the case of $\lambda = 0$, the feedback system is nonsmooth whilst, in the case of $\lambda > 0$, the tracking is approximate. In the context of linear systems, adaptive strategies—which incorporate internal models—for tracking constant signals or signals that are finite superpositions of sinusoidal functions may be found in [5,10]: we reiterate that the strategies developed in the present paper do not invoke the concept of an internal model and can handle reference signals of greater generality.

In the case $\lambda > 0$, the phrase λ-tracking has been used in, for example [1,3,6–9] to describe the associated control objective. Each of the latter references consider the λ-tracking problem ($\lambda > 0$) for multi-input, multi-output systems under a multivariable counterpart of the assumption that the sign of the constant b is known. By contrast, here we consider only the single-input, single-output case, but with $b \neq 0$ of unknown sign and with $\lambda \geqslant 0$: we stress that the case $\lambda = 0$, that of exact asymptotic tracking, is included within our framework; moreover, the class of admissible nonlinearities g and operators $\hat{T}$ is of considerable breadth.

The proposed strategy also tolerates an *output measurement disturbance* $v \neq 0$, provided that the disturbance belongs to the same function class as the reference signals. The disturbed error signal is then $e = (y + v) - r = y - (r - v)$. Therefore, from a strictly analytical viewpoint, in the presence of output disturbances of class $W^{1,\infty}(\mathbb{R})$, the disturbance-free analysis is immediately applicable on replacing the reference signal r by the signal $r - v$. Even though the reference signal r and disturbance signal v are assumed to be of the same class, practically, these signals might be distinguished by their respective spectra (v typically having “high-frequency” content). Moreover, from a practical viewpoint, one might reasonably expect that the disturbance v is “small”. In the presence of measurement disturbance, the adaptive control causes the disturbed output $\hat{y} = y + v$ to (approximately) track the reference r and so the true output y tracks the signal $r - v$ with asymptotic error quantified by λ. Therefore, if an a priori bound on the magnitude of the disturbance v is available, then a value of $\lambda > 0$ commensurate with that bound would be appropriate.

2. The system class $\mathcal{N}$

We consider single-input ($u(t)$), single-output ($y(t)$) systems described by nonlinear functional differential equations, suitably initialized, of the form

$$\dot{y}(t) = g(p(t), y(t)) + (\hat{T}y)(t) + bu(t), \tag{1}$$

where p is a perturbation term and $\hat{T}$ is a causal operator. This structure has, as prototype, the well-studied class of finite-dimensional, single-input, single-output, linear, minimum-phase systems (A, B, C) of relative degree one. In suitable coordinates, every such system takes the form

$$\dot{z} = \hat{A}z + \hat{B}y, \quad \dot{y} = \hat{C}z + \hat{D}y + CBu,$$

$$CB \neq 0, \quad (y(0), z(0)) = (y^0, z^0) \tag{2}$$

for some quadruple $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ where (by the minimum-phase assumption) the spectrum of $\hat{A}$ lies in the open left-half complex plane. Writing $b = CB \neq 0$, $g : (p, y) \mapsto p + \hat{D}y$ and defining the function p and the operator $\hat{T}$ by

$$p(t) = \hat{C}(\exp \hat{A}t)z^0,$$

$$(\hat{T}y)(t) = \hat{C} \int_0^t (\exp \hat{A}(t-s))\hat{B}y(s)\,\mathrm{d}s, \tag{3}$$

yields the equivalent form (1).

Before making precise the system class $\mathcal{N}$, we introduce some notation. For $I \subset \mathbb{R}$ an interval, $C(I; \mathbb{R}^N)$ (respectively, $AC_{\mathrm{loc}}(I; \mathbb{R}^N)$) denotes the set of continuous (respectively, locally absolutely continuous) functions $I \to \mathbb{R}^N$; $L^\infty(I; \mathbb{R}^N)$ is the Banach space of essentially bounded functions $x : I \to \mathbb{R}^N$ with the essential supremum norm $\| \cdot \|_\infty$; $L^\infty_{\mathrm{loc}}(I; \mathbb{R}^N)$ denotes the space of locally essentially bounded functions $\mathbb{R} \to \mathbb{R}^N$; if $N = 1$, then we simply write $C(I)$, $L^\infty(I)$ and $L^\infty_{\mathrm{loc}}(I)$. For $x : I \to \mathbb{R}^N$, the restriction of x to $J \subset I$ is denoted by $x|_J$, an extension of x to $\mathbb{R}$ is denoted by x^e. We write $\mathbb{R}_+ := [0, \infty)$. $\mathcal{K}$ denotes the class of continuous and strictly increasing functions $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$, with $\gamma(0) = 0$. Let

$$\mathcal{J} := \{\psi \in \mathcal{K} \,|\, \forall \delta \in \mathbb{R}_+ \ \exists \Delta \in \mathbb{R}_+ :$$

$$\psi(\delta s) \leqslant \Delta \psi(s) \ \forall s \in \mathbb{R}_+\}. \tag{4}$$

For example, (a) for each $q > 0$, the function $s \mapsto s^q$ is of class $\mathcal{J}$, (b) the function $s \mapsto \ln(1 + s)$ is of class $\mathcal{J}$; its inverse $s \mapsto \exp(s) - 1$ is of class $\mathcal{K}$ but is *not* of class $\mathcal{J}$.

Let $\mathcal{H}$ denote the following set of functions $\mathbb{R}_+ \to \mathbb{R}_+$:

$$\mathcal{H} := \{\varphi : \mathbb{R}_+ \to \mathbb{R}_+ \,|\, \varphi \text{ continuous and nondecreasing,}$$

$$\forall R > 0 \ \exists \mu_R > 0 : \varphi(|s + r|) \leqslant \mu_R \varphi(|s|)$$

$$\forall (s, r) \in \mathbb{R} \times [-R, R]\}. \tag{5}$$

Note that, if $\varphi \in \mathcal{H}$, then $\varphi(0) > 0$. As concrete examples, both $s \mapsto \exp(s)$ and $s \mapsto 1 + s^n$ $(n \in \mathbb{N})$ are in $\mathcal{H}$. Next, we introduce a class of operators.

Definition 1 (An operator class). For $h \geqslant 0$ and $\psi \in \mathcal{J}$, let $\mathcal{T}_h(\psi)$ denote the space of operators $T : L^\infty_{\mathrm{loc}}(\mathbb{R}) \to L^\infty_{\mathrm{loc}}(\mathbb{R})$ with the following properties:

1. (a) T is bounded-input, bounded-output stable in the following sense:

$$\forall R > 0 \ \exists M > 0 : \forall y \in L^\infty(\mathbb{R}), \ \|y\|_\infty < R$$

$$\Rightarrow |(Ty)(t)| < M \text{ for a.a. } t \in \mathbb{R}_+. \tag{6}$$

   (b) For some constant $\rho > 0$ the following holds: for each $y \in L^\infty_{\mathrm{loc}}(\mathbb{R})$, there exists a constant $\gamma > 0$ such that

$$\int_0^t |(Ty)(s)|\,|y(s)|\,\mathrm{d}s$$

$$\leqslant \gamma + \rho \int_0^t \psi(|y(s)|)|y(s)|\,\mathrm{d}s \quad \forall t \geqslant 0. \tag{7}$$

2. For all $t \geqslant 0$, the following hold:

   (a) for all $y, \xi \in L^\infty_{\mathrm{loc}}(\mathbb{R})$,

$$y(s) = \xi(s) \text{ for a.a. } s \in [-h, t]$$

$$\Rightarrow (Ty)(s) = (T\xi)(s) \text{ for a.a. } s \in [0, t];$$

   (b) for all $\zeta \in C([-h, t])$, there exist $\tau > 0$, $r > 0$ and $c > 0$ such that, for all $y, \xi \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ with $y|_{[-h,t]} = \zeta = \xi|_{[-h,t]}$ and $y(s), \xi(s) \in \mathbb{B}_r(\zeta(t))$ for all $s \in [t, t + \tau]$,

$$\text{ess-sup}_{s \in [t, t+\tau]} |(Ty)(s) - (T\xi)(s)|$$

$$\leqslant c \ \text{ess-sup}_{s \in [t, t+\tau]} |y(s) - \xi(s)|.$$

3. There exist constants $\delta, \mu > 0$ such that, for all $y, \xi \in L^\infty_{\mathrm{loc}}(\mathbb{R})$

$$|(T(y + \xi))(t)|$$

$$\leqslant \mu(|(T(\delta y))(t)| + |(T(\delta \xi))(t)|)$$

$$\text{for a.a. } t \in \mathbb{R}. \tag{8}$$

Remark 2.

(1) For the linear operator $\hat{T}$, arising from the finite-dimensional minimum-phase prototype

given by (3), Properties 1(a) and 1(b) hold with $\psi:\xi \mapsto \xi$ (an extension of this observation to an infinite-dimensional setting is given in Section 2.1(a)). Nonlinear choices of ψ facilitate certain cases of *nonlinear* operators T: examples of admissible nonlinear delay operators are given in Section 2.1(b).

(2) The essence of Property 2(a) of Definition 1 is that every $T \in \mathscr{T}_h$ is causal and has memory which extends back only to $-h$.

(3) Let $t \geqslant 0$. Given $y \in L^\infty([-h,t))$ let y^e denote an extension of y to $L^\infty_{\text{loc}}(\mathbb{R})$. By virtue of Property 2(a), $Ty^e|_{[0,t)}$ is uniquely determined by the function y in the sense that, the former is independent of the extension y^e chosen for the latter. Expanding on this observation, we will adopt the following notational convention. For $s \in [0,t)$, we simply write $(Ty)(s)$ in place of $(Ty^e)(s)$ (where y^e is any class $L^\infty_{\text{loc}}(\mathbb{R})$ extension of y).

(4) If $h_1 > h_2$, then $\mathscr{T}_{h_1}(\psi) \supset \mathscr{T}_{h_2}(\psi)$.

(5) Property 2(b) is a weak Lipschitz-type condition on T: this is a technical condition required to ensure that an appropriate existence theory [11, Theorem 2] can be invoked in the analysis of the closed-loop system.

(6) In comparison with [11], Property 3 is an additional condition imposed on T which enables the extension of the stabilization results in [11] to the tracking problem. Clearly, if T is linear, then Property 3 holds with $\mu = 1 = \delta$. Example (b) of Section 2.1 contains nonlinear operators T with Property 3.

We are now in a position to define the underlying system class.

Definition 3 (The system class). For $\varphi \in \mathscr{K}$, $\psi \in \mathscr{J}$ and $h \geqslant 0$, $\mathscr{N}(\varphi,\psi,h)$ is the class of all systems $(b,g,p,\hat T)$ of form (1) for which the following hold:

1. $b \neq 0$.
2. $g:\mathbb{R}^P \times \mathbb{R} \to \mathbb{R}$ is continuous and, for each compact $K \subset \mathbb{R}^P$ there exists $\mu_K \geqslant 0$:

$$|g(p,y)| \leqslant \mu_K \varphi(|y|) \quad \forall (p,y) \in K \times \mathbb{R}. \tag{9}$$

3. $p \in L^\infty(\mathbb{R};\mathbb{R}^P)$.
4. $\hat T = \sum_{i=1}^m T_i$, with $\hat T_i \in \mathscr{T}_h(\psi)$, $i = 1,\dots,m$.

2.1. Examples

(a) *Systems with linear infinite-dimensional subsystems* Σ_2: Let id denote the identity map on $\mathbb{R}_+$. Referring to [11, Section 3.3.2] for full details, assume that (A,B,C,D) are the generating operators of a regular linear system [13] with bounded observation operator C and with state space H (a Hilbert space). Define the operator $\hat T : L^\infty_{\text{loc}}(\mathbb{R}) \to L^\infty_{\text{loc}}(\mathbb{R})$ by

$$(\hat T y)(t) = \begin{cases} C\displaystyle\int_0^t S_{t-s} B y(s)\,\mathrm{d}s, & \text{a.a. } t \in \mathbb{R}_+, \\ 0 & \forall t < 0, \end{cases}$$

where $S = (S_t)_{t \in \mathbb{R}_+}$ is the semigroup generated by A. Assume that the semigroup S is exponentially stable. Then $\hat T \in \mathscr{T}_0(\text{id})$. Assume, in addition, that $b \neq 0$, $p \in L^\infty(\mathbb{R})$ and g is such that Property 9 of Definition 3 holds with $\varphi: s \mapsto 1 + s^3$. Then $(b,g,p,\hat T) \in \mathscr{N}(\varphi,\text{id},0)$.

For example, the following interconnection of a diffusion process Σ_2 (with spatial domain $[0,1]$, Dirichlet boundary conditions and, for notational simplicity, zero initial conditions) and a controlled ordinary differential equation Σ_1 corresponds to an admissible system of class $\mathscr{N}(\varphi,\text{id},0)$ for all $p_0, p_1, p_2, p_3 \in L^\infty(\mathbb{R})$, $a > 0$, $b \neq 0$, $0 < \xi_0 < \xi_i < 1$ and all $\varepsilon > 0$ sufficiently small:

$$\left.\begin{aligned} \dot y(t) &= p_0(t) + p_1(t)y(t) + p_2(t)y^{1/3}(t) \\ &\quad + p_3(t)y^3(t) + c\int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} z(t,\xi)\,\mathrm{d}\xi \\ &\quad + bu(t), \\ z_t(t,\xi) &= a z_{\xi\xi}(t,\xi) + \delta(\xi - \xi_i)y(t), \\ z(t,0) &= 0 = z(t,1) \quad \forall t \geqslant 0, \\ z(0,\xi) &= 0 \quad \forall \xi \in [0,1], \end{aligned}\right\} \tag{10}$$

where δ is the Dirac delta function.

(b) *Nonlinear systems with delay*: Let continuous ψ be such that, for some $\alpha > 1$, $\psi^{\alpha-1}(s) \geqslant s$ for all $s \in \mathbb{R}_+$.

Let $\Psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be such that (i) for each fixed v, $\Psi(\cdot,v)$ is measurable, and (ii) for every compact

$K \subset \mathbb{R}$, there exists a constant k such that

$$|\Psi(t,v)-\Psi(t,w)| \leqslant k|v-w| \quad \text{for a.a. } t \in \mathbb{R} \text{ and all } v,w \in K.$$

Assume further that (iii) for almost all t, the map $v \mapsto |\Psi(t,v)|$ is nonincreasing on $(-\infty,0)$ and nondecreasing on $(0,\infty)$, and (iv) for some constant $c>0$,

$$|\Psi(t,v)|^{\alpha} \leqslant c\psi(|v|)|v| \quad \text{for a.a. } t \text{ and all } y.$$

For example, if $n \in \mathbb{N}$, $\psi : s \mapsto 1+s^n$, $\alpha = 1+(1/n)$, $q \in L^{\infty}(\mathbb{R})$ and $1 \leqslant s \leqslant n$, then a map Ψ such that $|\Psi(t,v)| = |q(t)|\,|v|^s$ for almost all $t \in \mathbb{R}$ and all $v \in \mathbb{R}$ has properties (i)–(iv).

Now consider the operators $\hat{T}_{\mathrm{p}}$ and $\hat{T}_{\mathrm{d}}$ (point and distributed delays) defined on $L^{\infty}_{\mathrm{loc}}(\mathbb{R})$ by

$$\hat{T}_{\mathrm{p}}y := \Psi(\cdot, y(\cdot - h))$$

and

$$\hat{T}_{\mathrm{d}}y := \int_{-h}^{0} \Psi(s, y(\cdot + s))\,\mathrm{d}s.$$

Arguing as in [11, Section 3.2.1], it may be seen that Properties 1(a) and 1(b) of Definition 1 hold for $\hat{T}_{\mathrm{p}}$ and $\hat{T}_{\mathrm{d}}$. Moreover, by Property (iii) of Ψ, we may conclude that, for almost all $t \in \mathbb{R}$, $|\Psi(t,v+w)| \leqslant |\Psi(t,2v)| + |\Psi(t,2w)|$ for all $v,w \in \mathbb{R}$ and so Property 3 of Definition 1 also holds for $\hat{T}_{\mathrm{p}}$ and $\hat{T}_{\mathrm{d}}$. Therefore, $\hat{T}_{\mathrm{p}}, \hat{T}_{\mathrm{d}} \in \mathscr{T}_h(\psi)$.

As a concrete example, if $\psi = \varphi : s \mapsto 1+s^3$, then the following delay equation:

$$\begin{aligned}\dot{y}(t) &= p_0(t) + p_1(t)y(t) + p_2(t)y^{1/3}(t) + p_3(t)y^3(t)\\ &\quad + q_1(t)y(t-h_1) + q_2(t)y^2(t-h_2)\\ &\quad + \int_{-h_3}^{0} q_3(s)y^3(t+s)\,\mathrm{d}s + bu(t)\end{aligned} \tag{11}$$

with $p_i, q_i \in L^{\infty}(\mathbb{R})$ and $b \neq 0$, is a system of class $\mathscr{N}(\varphi,\psi,h)$, with $h = \max\{h_1,h_2,h_3\}$.

3. Universal servomechanism

Let $\psi \in \mathscr{H}$, $\varphi \in \mathscr{J}$, $h \geqslant 0$ and $\lambda \geqslant 0$. Define $\Lambda := [-\lambda,\lambda]$. The problem to be addressed is that of feedback control to ensure that, for every system of class $\mathscr{N}(\varphi,\psi,h)$ and arbitrary reference signal $r \in \mathscr{R} := W^{1,\infty}(\mathbb{R})$, the tracking error $e(t) := y(t) - r(t)$ approaches the set Λ, that is, $d_{\Lambda}(e(t)) \to 0$ as $t \to \infty$, where d_{Λ} is distance function for Λ given by

$$d_{\Lambda}(e) := \max\{0, |e|-\lambda\}.$$

Let $e \mapsto s_{\lambda}(e) \subset \mathbb{R}$ be the set-valued map defined on $\mathbb{R}$ as follows:

$$s_{\lambda} : e \mapsto \begin{cases}\{+1\}, & e > \lambda,\\ [-1,1], & |e| \leqslant \lambda,\\ \{-1\}, & e < -\lambda.\end{cases} \tag{12}$$

Define the continuous map

$$\Phi : \mathbb{R} \to \mathbb{R}_+, \quad e \mapsto \max\{\psi(d_{\Lambda}(e)), \varphi(|e|)\} \tag{13}$$

and consider the adaptive control strategy

$$\left.\begin{aligned}&u(t) \in \nu(k(t))\Phi(e(t))s_{\lambda}(e(t)),\\ &\dot{k}(t) = \Phi(e(t))d_{\Lambda}(e(t)), \quad k(0) = k^0,\end{aligned}\right\} \tag{14}$$

where $\nu \in C(\mathbb{R})$ has the Nussbaum properties

$$\text{(a)} \quad \limsup_{\eta\to\infty} \frac{1}{\eta}\int_0^{\eta} \nu = +\infty,$$

$$\text{(b)} \quad \liminf_{\eta\to\infty} \frac{1}{\eta}\int_0^{\eta} \nu = -\infty. \tag{15}$$

For example, $\nu : w \mapsto w^2 \cos w$ suffices.

Remark 4. If $\lambda > 0$, then, in implementing control (14), one can replace the set-valued map s_{λ} by any of its continuous selections (an obvious candidate being the simple saturation function sat_{λ} given by $\mathrm{sat}_{\lambda}(e) = e/\lambda$ for $|e| \leqslant \lambda$ and $\mathrm{sat}_{\lambda}(e) = \mathrm{sgn}(e)$ for $|e| > \lambda$). The set-valued nature of the feedback in (14) is a technical artifice to encompass, in a single analytical framework, both cases of $\lambda > 0$ (in which case the control is, in effect, continuous) and $\lambda = 0$ (in which case the control is unavoidably discontinuous, necessitating a set-valued interpretation).

Before analyzing system behaviour under the proposed feedback (14), we provide some intuition by highlighting a fundamental *high-gain property* of the system class. For simplicity, consider the case of $\lambda > 0$. With reference to Fig. 1, suppose that the control is given by a constant gain feedback

$$u(t) = -k\,\mathrm{sgn}(b)\Phi(e(t))\,\mathrm{sat}_{\lambda}(e(t)). \tag{16}$$

It can be shown that there exists a value $k^* > 0$ such that, for every fixed gain $k \geqslant k^*$, the feedback-controlled initial-value problem (1)–(16) is such that all signals are bounded and $d_\Lambda(e(t)) \to 0$ as $t \to \infty$. Realization of this constant-gain strategy requires knowledge of the sign of $b \neq 0$ and availability of sufficient plant data for the computation of a suitable threshold value k^*: in our general setting, such plant information is unavailable to the controller. In the adaptive strategy (14), the intuitive rôle of the function v includes that of compensating for lack of knowledge of the sign of b by the provision of a mechanism for generating gains of alternating polarity. The differential equation in (14) generates a non-decreasing function $k(\cdot)$ which, in composition with v, obviates the need to compute a threshold value k^* by providing an implicit mechanism for generating gain functions taking values of sufficiently large magnitude.

We now proceed to an analysis of system behaviour under the adaptive feedback (14).

Let $(b, g, p, \hat{T}) \in \mathcal{N}(\varphi, \psi, h)$ and $r \in \mathcal{R}$. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (with $f(\cdot, e)$ measurable and $f(t, \cdot)$ continuous) be such that

$$f(t,e) = g(p(t), e + r(t)) - \dot{r}(t)$$
$$\text{for a.a. } t \in \mathbb{R} \text{ and all } e \in \mathbb{R}.$$

Since $p \in L^\infty(\mathbb{R}; \mathbb{R}^P)$ and invoking Property 2 (Definition 3) of g, together with the facts that $r \in W^{1,\infty}(\mathbb{R})$ and $\varphi \in \mathscr{H}$ (recall (5)), we may infer the existence of constants $\hat{\mu}$ and $\tilde{\mu}$ such that

$$\begin{aligned} &|g(p(t), e + r(t))| + |\dot{r}(t)| \\ &\quad \leqslant \hat{\mu}\varphi(|e + r(t)|) + \|\dot{r}\|_\infty \\ &\quad \leqslant \tilde{\mu}(\varphi(|e|) + \varphi(0)) \\ &\qquad \text{for a.a. } t \in \mathbb{R} \text{ and all } e \in \mathbb{R}. \end{aligned}$$

Recalling that φ is non-decreasing and defining $\mu := 2\tilde{\mu}$, we have

$$|f(t,e)| \leqslant \mu\varphi(|e|) \text{ for a.a. } t \in \mathbb{R} \text{ and all } e \in \mathbb{R}.$$

Let $\hat{T}_r$ be defined by $(\hat{T}_r x)(\cdot) := (\hat{T}(x + r))(\cdot)$ for all $x \in L^\infty_{\text{loc}}(\mathbb{R})$.

Now consider (1) with initial data $y_{[-h,0]} = y^0 \in C([-h, 0]; \mathbb{R})$: in terms of the error, the initial-value problem may be expressed as

$$\left.\begin{aligned} &\dot{e}(t) - (\hat{T}_r e)(t) = f(t, e(t)) + bu(t) \\ &e|_{[-h,0]} = (y^0 - r)(\cdot) \in C([-h, 0]; \mathbb{R}). \end{aligned}\right\} \qquad (17)$$

Define the set-valued map F by

$$\begin{aligned} F(x) &= F(e,k) \\ &:= \{v + bu\colon |v| \leqslant \mu\varphi(|e|), u \in v(k)\Phi(e)s_\lambda(e)\} \\ &\qquad \times \{\Phi(e)d_\Lambda(e)\}. \end{aligned}$$

Note that F is upper semicontinuous with nonempty, compact, convex values in $\mathbb{R}^2$. Extend $k(s)$ for $s < 0$ by defining $k(s) = k^0$ for all $s < 0$. Under feedback (14), the initial-value problem for the closed-loop system may be embedded in the following initial-value problem

$$\left.\begin{aligned} &\dot{x}(t) - (Tx)(t) \in F(x(t)), \\ &x|_{[-h,0]} = x^0 := ((y^0 - r)(\cdot), k^0) \\ &\quad \in C([-h, 0]; \mathbb{R}^2), \end{aligned}\right\} \qquad (18)$$

where $x(t) = (e(t), k(t))$ and the operator $T : L^\infty_{\text{loc}}(\mathbb{R}; \mathbb{R}^2) \to L^\infty_{\text{loc}}(\mathbb{R}; \mathbb{R}^2)$ is given by

$$Tx = T(e,k) = (\hat{T}_r e, 0).$$

By a *solution* of (18) we mean a function $x \in C([-h, \omega); \mathbb{R}^2)$ for some $\omega > 0$, such that $x|_{[-h,0]} = x^0(\cdot)$, and $x|_{[0,\omega)} \in AC_{\text{loc}}([0, \omega); \mathbb{R}^2)$ with $\dot{x}(t) - (Tx)(t) \in F(x(t))$ for almost all $t \in [0, \omega)$. A solution is said to be *maximal* if it does not have a proper right extension which is also a solution. Now, F and T are such that the hypotheses of the existence Theorem 2 in [11] hold, and so we conclude the following.

Lemma 5. *There exists a solution of the initial-value problem* (18) *on some interval* $[-h, \omega)$ *with* $\omega > 0$ *and every solution can be maximally extended*; *moreover, if a maximal solution* $x : [-h, \omega) \to \mathbb{R}^2$ *is bounded, then* $\omega = \infty$.

We now arrive at the main result which asserts that (14) is a $(\mathcal{N}(\varphi, \psi, h), \mathcal{R})$-universal servomechanism.

Theorem 6. *Let* $x = (e,k) : [-h, \omega) \to \mathbb{R}^2$ *be a maximal solution of* (18). *Then* (i) $\omega = \infty$, (ii) $\lim_{t\to\infty} k(t)$

exists and is finite, and (iii) $d_\Lambda(e(t)) \to 0$ *as* $t \to \infty$.

Proof. Introduce the continuously differentiable function $V:\mathbb{R} \to \mathbb{R}_+$, $z \mapsto \frac{1}{2}d_\Lambda^2(z)$, with derivative DV given by

$$DV(z) = \begin{cases} d_\Lambda(z)\,\mathrm{sgn}(z), & |z| > \lambda, \\ 0, & |z| \leqslant \lambda. \end{cases}$$

For all $t \in [-h,\omega)$, define $\zeta(t) := DV(e(t))$ and note that $t \mapsto e(t) - \zeta(t) + r(t)$ is bounded on $[-h,\omega)$.

By Property 4 of Definition 3 and Property 3 of Definition 1, there exist constants c, $\delta_i > 0$ and operators $T_i \in \mathscr{T}_h(\psi)$ such that

$$\begin{aligned}|(\hat{T}_r e)(t)| &\leqslant \sum_{i=1}^m |(T_i(e+r))(s)| \\ &\leqslant c\sum_{i=1}^m [|(T_i\delta_i\zeta)(s)| + |T_i(\delta_i(e-\zeta+r))(s)|], \\ &\quad \text{a.a. } t \in [0,\omega).\end{aligned}$$

By Property 1(a) of Definition 1, together with boundedness of $e - \zeta + r$, monotonicity of φ and recalling that $\varphi(0) > 0$, there exists constant c_0 such that

$$|(\hat{T}_r e)(t)| \leqslant c_0\left(\varphi(|e(s)|) + \sum_{i=1}^m |(T_i(\delta_i\zeta))(s)|\right), \quad \text{a.a. } t \in [0,\omega). \tag{19}$$

Invoking Property 1(b) of Definition 1, we may infer that, for $1 \leqslant i \leqslant m$, there exist constants ρ_i and functions γ_i such that

$$\begin{aligned}\int_0^t |T_i(\delta_i\zeta)(s)|\,|\zeta(s)|\,\mathrm{d}s &= \frac{1}{\delta_i}\int_0^t |T_i(\delta_i\zeta)(s)|\,|\delta_i\zeta(s)|\,\mathrm{d}s \\ &\leqslant \frac{\hat{\gamma}_i}{\delta_i} + \rho_i\int_0^t \psi(\delta_i d_\Lambda(s))d_\Lambda(s)\,\mathrm{d}s, \\ &\quad \forall t \in [0,\omega),\end{aligned}$$

where $\hat{\gamma}_i = \gamma_i(\delta_i\zeta^e)$, ζ^e being the extension of ζ to $\mathbb{R}$ given by $\zeta^e(t) = \zeta(t)$ for $t \in [-h,\omega)$ and $\zeta^e(t) = 0$ for $t \in \mathbb{R}\setminus[-h,\omega)$. Since $\psi \in \mathscr{J}$ (recall (4)), there exist constants Δ_i such that $\psi(\delta_i d_\Lambda(t)) \leqslant \Delta_i\psi(d_\Lambda(t))$ for all $t \in [0,\omega)$, $i = 1,\ldots,m$. Therefore, we may conclude the existence of a constant $c_1 \geqslant 0$, such that

$$\begin{aligned}\int_0^t |T_i(\delta_i\zeta)(s)|\,|\zeta(s)|\,\mathrm{d}s &\leqslant c_1\left(1 + \int_0^t \psi(d_\Lambda(e(s)))d_\Lambda(e(s))\,\mathrm{d}s\right) \\ &\quad \forall t \in [0,\omega),\ i = 1,\ldots,m.\end{aligned} \tag{20}$$

Combining (19) and (20) and recalling definition (13) of Φ, it follows that there exists $c_2 \geqslant 0$ such that

$$\begin{aligned}\int_0^t |\hat{T}_r(e(s))|\,|\zeta(s)|\,\mathrm{d}s &\leqslant c_2\left(1 + \int_0^t \Phi(e(s))d_\Lambda(e(s))\,\mathrm{d}s\right) \\ &= c_2\left(1 + \int_0^t \dot{k}(s)\,\mathrm{d}s\right) \\ &= c_2(1 + k(t) - k^0) \quad \forall t \in [0,\omega).\end{aligned} \tag{21}$$

Note that, from (14),

$$uDV(e(s)) = u\zeta(s) = v(k(s))\dot{k}(s) \quad \forall u \in v(k(s))\Phi(e(s))s_\lambda(e(s)) \quad \forall s \in [0,\omega).$$

Therefore,

$$\begin{aligned}(V \circ e)'(t) &= DV(e(t))\dot{e}(t) = \zeta(t)\dot{e}(t) \\ &\leqslant \mu|\zeta(t)|\varphi(|e(t)|) + |\zeta(t)|\,|(\hat{T}_r e)(t)| + bv(k(t))\dot{k}(t) \\ &\leqslant \mu\dot{k}(t) + |\zeta(t)|\,|(\hat{T}_r e)(t)| + bv(k(t))\dot{k}(t), \\ &\quad \text{a.a. } t \in [0,\omega),\end{aligned} \tag{22}$$

which, on integration and invoking (21), yields

$$\begin{aligned}0 &\leqslant V(e(t)) \\ &\leqslant V(e(0)) + c_2 + c_3(k(t) - k^0) + b\int_{k^0}^{k(t)} v(s)\,\mathrm{d}s \\ &\quad \text{for a.a. } t \in [0,\omega),\end{aligned} \tag{23}$$

where $c_3 = c_2 + \mu$.

Seeking a contradiction, suppose k is unbounded and so, by monotonicity, $k(t) \to \infty$ as $t \uparrow \omega$. Let $\tau \in [0,\omega)$ be such $k(\tau) > 0$. From (23), we infer the existence of a constant $c_4 > 0$ such that

$$0 \leqslant c_4 + \frac{b}{k(t)}\int_{k(\tau)}^{k(t)} v(s)\,\mathrm{d}s \quad \forall t \in [\tau,\omega). \tag{24}$$

Recalling that $b \neq 0$ and taking limit superior (if $b < 0$) or limit inferior (if $b > 0$) as $t \uparrow \omega$ in (24), we contradict one or the other of properties (15) of v. Therefore, k is bounded.

By (23), it follows that $V \circ e$ is bounded. Hence $d_\Lambda(e(\cdot))$ is bounded and so e is bounded. We have now shown that $x=(e,k)$ is bounded and so (by Lemma 5) $\omega=\infty$. This proves assertion (i) of the theorem. Assertion (ii) follows by boundedness and monotonicity of k. It remains to prove assertion (iii). By boundedness of x, together with (18) and properties of F and T, $\dot{x} \in L^\infty(\mathbb{R}_+)$ and so x is uniformly continuous. By boundedness and uniform continuity of e, it follows that $l:\mathbb{R}_+ \to \mathbb{R}_+, t \mapsto \Phi(e(t))d_\Lambda(e(t))$ is uniformly continuous. Moreover, by boundedness of k

$$\infty > \int_0^\infty \dot{k}(s)\,\mathrm{d}s = \int_0^\infty l(s)\,\mathrm{d}s.$$

By Barbălat's Lemma [2], we may conclude that $d_\Lambda(e(t)) \to 0$ as $t \to \infty$. □

4. Examples revisited

Let $\varphi = \psi : s \mapsto 1 + s^3$ and consider again the two systems (10) and (11) of Section 2.1 with $h=\max\{h_1,h_2,h_3\}$. Noting that $\mathscr{T}_0(\mathrm{id}) \subset \mathscr{T}_h(\psi)$, we see that each system is of class $\mathscr{N}(\varphi,\psi,h)$. Therefore, for every $r \in \mathscr{R}$ and $\lambda \geqslant 0$, the tracking objective is achieved for each of these two disparate systems by the control

$$u(t) \in v(k(t))[1+|e(t)|^3]s_\lambda(e(t)),$$

$$\dot{k}(s) = [1+|e(t)|^3]d_\Lambda(e(t)), \qquad k(0)=0. \tag{25}$$

By way of illustration, consider a specific case of system (11):

$$\dot{y}(t) = \sin(7t) + y^{1/3}(t) + y^3(t)$$
$$+ y(t-1) + y^2(t-2)$$
$$+ \int_{-2}^{0} \sin(-\pi s/2)y^3(t+s)\,\mathrm{d}s + u(t)$$

with initial data $y(t)=t/2+1$ for $-2 \leqslant t \leqslant 0$ and with output measurement disturbance $v \in W^{1,\infty}(\mathbb{R}_+)$ (see Fig. 1) given by $v(t)=\zeta_1(300t)/10$ for all $t \geqslant 0$, where ζ_1 is the first component of the (bounded, chaotic) solution $\zeta = (\zeta_1,\zeta_2,\zeta_3)$ of the initial-value problem

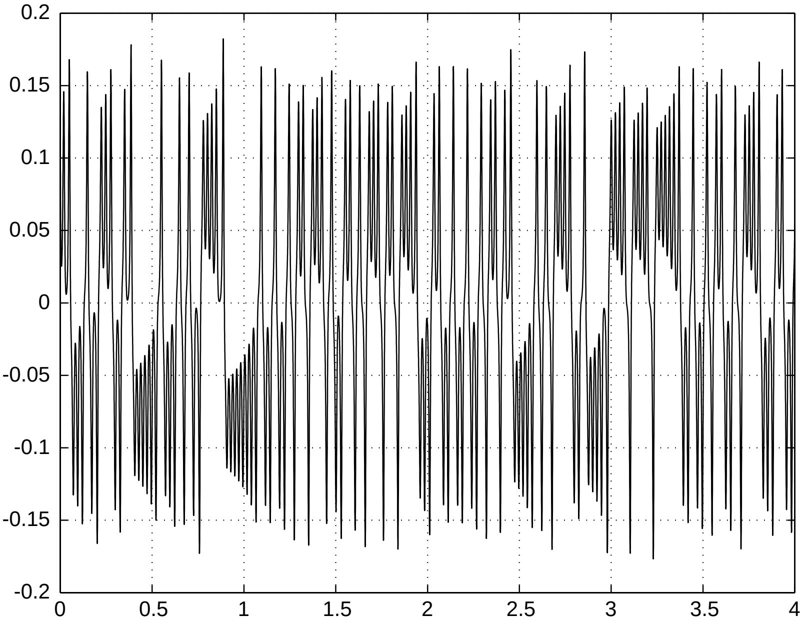

Fig. 2. Output measurement disturbance.

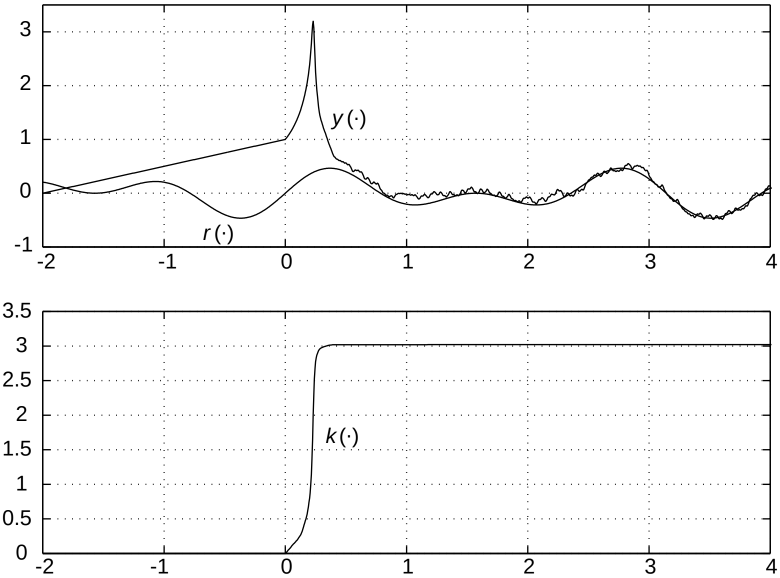

Fig. 3. Typical behaviour of the adaptively controlled system.

for the Lorenz system

$$\dot{\zeta}_1(t) = \zeta_2(t) - \zeta_1(t), \qquad \zeta_1(0) = 1,$$

$$\dot{\zeta}_2(t) = 2.8\zeta_1(t) - 0.1\zeta_2(t) - \zeta_1(t)\zeta_3(t), \qquad \zeta_2(0) = 0,$$

$$\dot{\zeta}_3(t) = \zeta_1(t)\zeta_2(t) - 8\zeta_3(t)/30, \qquad \zeta_3(0) = 3.$$

The measurement disturbance signal $t \mapsto v(t) = \zeta_1(300t)/10$ is depicted in Fig. 2.

With $r : t \mapsto \sin(5t)/4 + \sin(3t)/4$, $\lambda = \frac{1}{5}$, $v : w \mapsto w^2 \cos(w)$ and choosing the continuous selection for s_λ given by the simple saturation function sat_λ defined in Remark 4, the adaptively controlled system behaviour, in the presence of output disturbance v, is shown in Fig. 3 (computed using an improved Euler method within MATLAB).

Remark 7. The behaviour in Fig. 3 appears typical for the class of systems considered: an initial transient phase, during which the gain adapts, followed by a quiescent phase during which the tracking objective has essentially been achieved. Of course, the proof of Theorem 6 ensures only asymptotic attraction to the set $\Lambda = [-\lambda, \lambda]$; performance issues relating to transient behaviour are not addressed.

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