Group Theory: Sheet 1

1. Consider the group $G = \mathrm{GL}(n, \mathbb{C})$ of n by n invertible matrices with complex entries, the group operation being matrix multiplication. A subset S of $\mathrm{GL}(n, \mathbb{C})$ may or may not form a group in its own right (using the same group operation, and the same identity element I_n). Which of the following subsets of G form groups in their own right?

   (a) The elements of G which are symmetric matrices.

   (b) The elements $X \in G$ such that $X\bar{X}^T = I_n$. Here $\bar{X}$ is obtained from X by replacing each entry of X by its complex conjugate. We use superscript T to denote taking the transpose of a matrix.

   (c) The elements X in G which have *finite order*. (An element X has finite order if there is a positive integer m such that X^m is the identity element.)

2. Let G be the symmetric group $\mathrm{Sym}(\mathbb{Z})$, so G consist of all bijections from $\mathbb{Z}$ to $\mathbb{Z}$. The *order* n of an element x in a group is the smallest natural number (if there are any) such that $x^n = 1$. Note that the identity element has order 1, and it is possible that there are no natural numbers with the property – hence the notion of *infinite order* mentioned above.

   (a) Suppose that p, q are positive integers. Prove that there are elements $\theta, \psi \in G$ with θ of order p, ψ of order q, where θ commutes with ψ.

   (b) Suppose that p, q are positive integers. Prove that there are elements $\theta, \psi \in G$ with θ of order p, ψ of order q, but θ does not commute with ψ.

   (c) Give an example of an element $\zeta \in G$ of infinite order.

   (d) Give examples of elements $\alpha, \beta \in$, each of infinite order, such that $\alpha\beta$ has order 2.

   (e) Suppose that r is a positive integer bigger than 2. Do there exist $\mu, \nu \in G$ both of infinite order such that $\mu\nu$ has order r?

   (f) Do there exist $\sigma, \tau \in G$ both of infinite order such that σ, τ has infinite order?

   (g) Suppose that $\eta, \nu \in G$ both have finite order. Does it follow that $\eta\nu$ has finite order?

3. Let G be a group, and suppose that $x, y \in G$. Prove that xy and yx have the same order. This means that both have infinite order, or both have the same finite order.

4. Let G be a group, and suppose that $x, y, z \in G$. Prove that xyz, yzx and zxy must all have the same order. Compare this with the order of xzy.