Group Theory: Sheet 3

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The course web site is available via `http://www.bath.ac.uk/~masgcs/`

1. Do exercises 1.11, 1.12, 2.1 to 2.7 in *Topics in Group Theory*. Only look at the answers at the back of the book if you get really stuck.

2. Suppose that $\alpha : G \to H$ is an isomorphism of groups. Show that the map $\alpha^{-1} : H \to G$ is also an isomorphism.

3. (a) Suppose that α is an automorphism of a group G. Let $F_\alpha = \{g \in G \mid (g)\alpha = g\}$. Prove that $F_\alpha \leq G$.

   (b) Let $G = \mathrm{GL}(n, \mathbb{C})$ be the group of invertible n by n complex matrices. Consider the map $\beta : G \to G$ which sends each matrix to the transpose of its inverse. Prove that β is an automorphism of G. Deduce that the subset of G consisting of those $g \in G$ such that $gg^T = I_n$ is a group.

4. Determine all automorphisms of the additive group $\mathbb{Z}$. Determine all homomorphisms $\gamma : \mathbb{Z} \to \mathbb{Z}$.

5. Determine all automorphisms of the additive group $\mathbb{Q}$ of rational numbers.

6. List all the automorphisms of S_3, and demonstrate that you have found them all.

7. Suppose that H is a subgroup of a group G. Let $\Omega = G/H$ so Ω is a collection of subsets of G. Subsets of G may be multiplied together. Suppose that Ω is closed under multiplication. Prove that Ω is a group, and deduce that H is a normal subgroup of G.

8. Suppose that G is a group with subgroups H and K. Suppose that both H and K have finite index in G. Deduce that $H \cap K$ must have finite index in G.

9. Exhibit a group G with the following property: the intersection of all subgroups of finite index in G is a subgroup of infinite index in G.