Group Theory: Sheet 7

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1. Let G be a finite abelian group and p and prime number which divides $|G|$. Suppose that $|G| = p^n q$ where $p \not| q$.
   (a) Let $G(p) = \{g \in G \mid g^{p^n} = 1$. Prove that $G(p)$ is a subgroup of G.
   (b) Let $G(p') = \{g \in G \mid g^q = 1$ for some $q \in \mathbb{N}$ with $gcd(p, q) = 1\}$. Prove that $G(p')$ is a subgroup of G.
   (c) Prove that $G(p) \cap G(p') = 1$.
   (d) Suppose that $g \in G$ has order rs where r is a power of p and s is coprime to p. Recall that there must exist $\lambda, \mu \in \mathbb{Z}$ such that $\lambda r + \mu s = 1$. Prove that $g \in G(p)G(p')$ and deduce that $G = G(p)G(p')$.
   (e) Show that $|G(p)| = p^n$ and $|G(p')| = q$.
   (f) Deduce that G must contain an element of order p.

2. How many essentially different ways are there to paint the faces of a cube if two faces must be red, two white and two blue. *Two paintings are essentially the same if you can move from one to the other by means of a rigid motion. No reflections are allowed.*

3. You have 24 beads to thread on to a circular string to make a necklace. There are 8 blue beads, 6 red beads, 4 white beads, 2 black beads and 4 green beads. How many essentially different necklaces can you make?

4. How many essentially different ways are there to paint half the vertices of a regular dodecehedron black, and half white?

5. Let G be a finite group. It turns out that the average (over all group elements) size of the centralizer of an element must be an integer. Why?

6. Let the group G act on a finite non-empty set Ω of size at least 2. Let Γ denote the subset of Ω^2 consisting of those ordered pairs (ω_1, ω_2) where the entries are distinct. There is a natural action of G on Γ defined by $(\omega_1, \omega_2) \cdot g = (\omega_1 g, \omega_2 g)$ for all $(\omega_1, \omega_2) \in \Gamma$. Show that the following statements are equivalent.
   (a) G has just one orbit as it acts on Γ.

(b) G has just one orbit as it acts on Ω and there is $\omega \in \Omega$ such that G_ω has two orbits as it acts on Ω.

(c) G has just one orbit as it acts on Ω and every $\omega \in \Omega$ has the property that G_ω has two orbits as it acts on Ω.

7. Let G be a finite group and $m \in \mathbb{N}$. It turns out that the average (over all group elements) of the m-th power of the size of the centralizer of an element must be an integer. Why?