Group Theory: Sheet 9

The course web site is available via `http://www.bath.ac.uk/~masgcs/`

1. Suppose that $\alpha : G \to H$ is a group epimorphism. Suppose that $K \leq G$ and let $\overline{K}$ denote the group $\{(k)\alpha \mid k \in K\}$. Suppose that $L \leq H$. Let $\widehat{L}$ denote the group $\{g \in G \mid (g)\alpha \in L\}$. Let $N = \text{Ker } \alpha$.

   (a) Suppose that $A \leq G$. Show that $\widehat{\overline{A}} = AN$.

   (b) Suppose that $B \leq H$. Show that $\overline{\widehat{B}} = B$.

   (c) Suppose that $N \leq C \leq D \leq G$. Show that $|D : C| = |\overline{D} : \overline{C}|$ and that $C \trianglelefteq D$ if and only if $\overline{C} \trianglelefteq \overline{D}$.

   (d) Suppose that $E \leq F \leq H$. Show that $|F : E| = |\widehat{F} : \widehat{E}|$.

   (e) Show that "bar" and "hat" set up an inclusion preserving, index preserving bijection between the subgroups of G which contain N, and the subgroups of H.

2. Suppose that G is a finite group and that $N \trianglelefteq G$. Let p be a prime number and suppose that $P \in \text{Syl}_p(N)$. Show that $G = N_G(P)N$.

3. Suppose that G is a finite group and that p is a prime number. Let P be a Sylow p subgroup of G.

   (a) Suppose that $N_G(P) \leq H \leq G$. Prove that $N_G(H) = H$.

   (b) Suppose that $N_G(P) \leq K \leq G$ and that $N_G(P) \leq L \leq G$. Furthermore suppose that there is $g \in G$ such that $H^g = K$. Prove that $H = K$.