Nonlinear Analysis 71 (2009) 4769–4789

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Absolute stability and integral control for infinite-dimensional discrete-time systems☆

J.J. Coughlan, H. Logemann*

Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK

ARTICLE INFO

Article history:
Received 25 January 2008
Accepted 20 March 2009

MSC:
39A11
39A12
93B52
93C10
93C20
93C55
93D05
93D10
93D25

Keywords:
Absolute stability
Nonlinear Volterra difference equations
Circle criterion
Discrete-time systems
Infinite-dimensional systems
Integral control
Popov criterion
Positive-real

ABSTRACT

We derive absolute stability results of Popov and circle-criterion types for infinite-dimensional discrete-time systems in an input–output setting. Our results apply to feedback systems in which the linear part is the series interconnection of an l^2-stable linear system and an integrator and the nonlinearity satisfies a sector condition which allows for saturation and deadzone effects. The absolute stability theory is then used to prove tracking and disturbance rejection results for integral control schemes in the presence of input and output nonlinearities. Applications of the input–output theory to state-space systems are also provided.

1. Introduction

Absolute stability problems and their relations to positive-real conditions have played a prominent role in systems and control theory and have led to a number of important stability criteria for unity feedback controls applied to linear dynamical systems subject to static input or output nonlinearities, the Popov and circle criteria being the most prominent examples (see, e.g., [1–4] for the continuous-time and [1,3,5] for the discrete-time cases).

In this paper, we study an absolute stability problem for the discrete-time feedback system shown in Fig. 1. The input–output operator G is linear, shift-invariant and bounded from $l^2(\mathbb{Z}_+, U)$ into itself, $\varphi : \mathbb{Z}_+ \times U \to U$ is a (time-varying) nonlinearity (where U is a real Hilbert space) and J is the strictly causal discrete-time integrator given by

$$(Jv)(n) = \sum_{j=0}^{n-1} v(j), \quad \text{for functions } v : \mathbb{Z}_+ \to U.$$

☆ This work was supported in part by UK EPSRC Grant GR/S94582/01.

* Corresponding author. Tel.: +44 1225 386008; fax: +44 1225 386492.
E-mail addresses: jcoughlan88@hotmail.com (J.J. Coughlan), hl@maths.bath.ac.uk (H. Logemann).

0362-546X/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2009.03.072

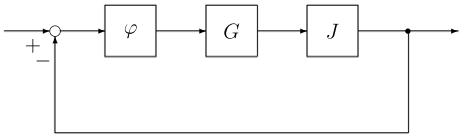

Fig. 1. Feedback system.

The two main stability theorems in Section 2 are a result of Popov type (Theorem 2.1) and a result of circle-criterion type (Theorem 2.2). The former is restricted to single-input–single-output systems (i.e., $U = \mathbb{R}$) and time-independent nonlinearities φ. In a sense, these results form the discrete-time counterparts of the continuous-time stability theorems obtained in [6]. We emphasize that in contrast to previous results in the literature on absolute stability of discrete-time systems (see, e.g., [1,3,5,7–10], most of which deal with finite-dimensional systems), the two main results in Section 2 (Theorems 2.1 and 2.2) consider feedback systems, where the linear part contains an integrator (meaning in particular that the linear system is not input–output stable) and where at the same time the lower gain $\inf_{v\in U} \|\varphi(v)\|/\|v\|$ of the nonlinearity φ is allowed to be equal to zero (which for example is the case for bounded nonlinearities such as saturation). One of the motivations for studying this situation is its importance in Section 3, where we use the absolute stability results from Section 2 to develop an input–output theory of low-gain integral control in the presence of (possibly saturating) input and/or output nonlinearities. The low-gain integral control problem has its roots in control engineering, where it is often required that the output y of a system tracks a constant reference signal ρ, i.e., the error $e(n) := y(n) - \rho$ should converge to 0 as $n \to \infty$. It is well known that for linear power-stable single-input–single-output systems with positive steady-state gain (i.e., $\mathbf{G}(1) > 0$, where $\mathbf{G}$ denotes the transfer function), this can be achieved by feeding the error into an integrator with sufficiently small positive gain parameter and then closing the feedback loop (see [11]). In Section 3, we develop generalizations of this result to linear systems subject to input and/or output nonlinearities. The main results in Section 3 (Theorems 3.2 and 3.4) constitute the discrete-time counterpart of the continuous-time low-gain integral control theory developed in [6,12]. We close the paper with Section 4 which is devoted to applications of the input–output results in Sections 2 and 3 to infinite-dimensional state-space systems.

In a forthcoming paper (see [13]), the authors will apply the discrete-time theory developed in this paper to low-gain sampled-data integral control of well-posed linear infinite-dimensional continuous-time systems subject to actuator and sensor nonlinearities.

Notation and terminology. Let X be a Banach space and U be a Hilbert space. We define $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{Z}_+ := \{0, 1, \ldots\}$. Let $F(\mathbb{Z}_+, X)$ denote the set of X-valued functions defined on $\mathbb{Z}_+$. We define functions δ and ϑ in $F(\mathbb{Z}_+, \mathbb{R})$ by

$$\delta(n) := \begin{cases} 1, & n = 0 \\ 0, & n \geq 1 \end{cases} \quad \text{and} \quad \vartheta(n) := 1, \quad \forall\, n \in \mathbb{Z}_+.$$

For $1 \leq p \leq \infty$, let $l^p(\mathbb{Z}_+, X)$ denote the l^p-space of unilateral X-valued sequences. The subspace of all functions $w \in F(\mathbb{Z}_+, X)$ which admit a decomposition of the form $w = w_1 + w_2\vartheta$, where $w_1 \in l^p(\mathbb{Z}_+, X)$ and $w_2 \in X$ is denoted by $m^p(\mathbb{Z}_+, X) := l^p(\mathbb{Z}_+, X) + X$. Endowed with the norm $\|w\|_{m^p} := \|w_1\|_{l^p} + \|w_2\|$, the space $m^p(\mathbb{Z}_+, X)$ is complete. In the special case $X = \mathbb{R}$, we set $F(\mathbb{Z}_+) := F(\mathbb{Z}_+, \mathbb{R})$, $l^p(\mathbb{Z}_+) := l^p(\mathbb{Z}_+, \mathbb{R})$ and $m^p(\mathbb{Z}_+) := m^p(\mathbb{Z}_+, \mathbb{R})$.

We define $\mathbb{E} := \{z \in \mathbb{C} : |z| > 1\}$, the exterior of the closed unit disc. Let $H^\infty(\mathbb{E}, X)$ denote the space of bounded holomorphic functions defined on $\mathbb{E}$ with values in X; $H^2(\mathbb{E}, U)$ denotes the Hardy–Lebesgue space of square-integrable holomorphic functions defined in $\mathbb{E}$ with values in U. We set $H^\infty(\mathbb{E}) := H^\infty(\mathbb{E}, \mathbb{C})$ and $H^2(\mathbb{E}) := H^2(\mathbb{E}, \mathbb{C})$. If $f \in H^2(\mathbb{E}, U)$, then the non-tangential limits

$$f(e^{i\theta}) := \lim_{z \to e^{i\theta},\ z\in\mathbb{E}} f(z)$$

exist for almost every $\theta \in [0, 2\pi)$ and the function $\theta \mapsto f(e^{i\theta})$ is in $L^2(\partial\mathbb{E}, U)$. Moreover, it is well known that a function $f : \mathbb{E} \to U$ belongs to $H^2(\mathbb{E}, U)$ if and only if there exists $u \in l^2(\mathbb{Z}_+, U)$ such that $\mathscr{Z}u = f$, where $\mathscr{Z}u$ denotes $\mathscr{Z}$-transform of u, that is $(\mathscr{Z}u)(z) = \sum_{j=0}^{\infty} u(j)z^{-j}$ for all $z \in \mathbb{E}$. We will frequently use the notation $\widehat{u} = \mathscr{Z}u$. The *Parseval–Bessel* identity says that, for $u, v \in l^2(\mathbb{Z}_+, U)$,

$$\sum_{j=0}^{\infty} \langle u(j), v(j)\rangle = \frac{1}{2\pi} \int_0^{2\pi} \langle \widehat{u}(e^{i\theta}), \widehat{v}(e^{i\theta})\rangle \mathrm{d}\theta.$$

We write $H^\infty(\mathbb{E}) := H^\infty(\mathbb{E}, \mathbb{C})$ and $H^2(\mathbb{E}) := H^2(\mathbb{E}, \mathbb{C})$.

For $z \in X$, a set $S_z \subset X$ is called a sphere centred at z if there exists $\eta \geq 0$ such that $S_z = \{v \in X : \|v - z\| = \eta\}$. For a function $f \in F(\mathbb{Z}_+, X)$ and a subset $V \subset X$, we say that $f(n)$ approaches V as $n \to \infty$ if $\mathrm{dist}(f(n), V) = \inf_{v\in V} \|f(n) - v\| \to 0$. For a set $V \subset X$, we denote by $\mathrm{cl}(V)$ the closure of V in X and by $\mathrm{int}(V)$ the interior of V. We use $\mathscr{B}(X_1, X_2)$ to denote the space of bounded linear operators from a Banach space X_1 to a Banach space X_2; we set $\mathscr{B}(X) := \mathscr{B}(X, X)$. The resolvent set of $A \in \mathscr{B}(X)$ is denoted by $\varrho(A)$.

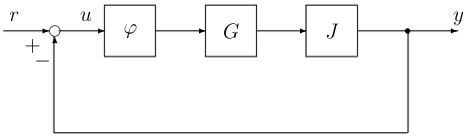

Fig. 2. Feedback system.

For $N \in \mathbb{Z}_+$, we define the projection $P_N : F(\mathbb{Z}_+, X) \to F(\mathbb{Z}_+, X)$ by $(P_N f)(n) = f(n)$ if $n \leq N$ and $(P_N f)(n) = 0$ if $n > N$. We define the *right-shift operator* $S : F(\mathbb{Z}_+, X) \to F(\mathbb{Z}_+, X)$ by $(Sv)(n) = 0$ if $n = 0$ and $(Sv)(n) = v(n-1)$ if $n \geq 1$. The *forward difference operator* $\Delta : F(\mathbb{Z}_+, X) \to F(\mathbb{Z}_+, X)$ is defined by $(\Delta v)(n) := v(n+1) - v(n)$, for all $n \in \mathbb{Z}_+$ and the *backward difference operator* $\nabla : F(\mathbb{Z}_+, X) \to F(\mathbb{Z}_+, X)$ is defined by $\nabla = S\Delta + \delta I$, that is, $(\nabla v)(n) := v(0)$ if $n = 0$ and $(\nabla v)(n) = v(n) - v(n-1)$ if $n \geq 1$. The *discrete-time integrator* $J : F(\mathbb{Z}_+, X) \to F(\mathbb{Z}_+, X)$ is defined by

$$(Jv)(n) := \begin{cases} 0, & n = 0 \\ \sum_{j=0}^{n-1} v(j), & n \geq 1. \end{cases}$$

2. Discrete-time absolute stability results

We consider an absolute stability problem for the feedback system shown in Fig. 2. The input–output operator $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$ is shift-invariant, that is, $SG = GS$, $\varphi : \mathbb{Z}_+ \times U \to U$ is a time-dependent nonlinearity and $r : \mathbb{Z}_+ \to U$ is the input of the feedback system (or forcing function). Since the operator G is shift-invariant, it is causal, i.e., $P_N G P_N = P_N G$ for all $N \in \mathbb{Z}_+$. By causality, G extends to a shift-invariant operator from $F(\mathbb{Z}_+, U)$ into itself. We shall use the same symbol G to denote the original operator on $l^2(\mathbb{Z}_+, U)$ and its shift-invariant extension to $F(\mathbb{Z}_+, U)$.

The feedback system shown in Fig. 2 is described by the equation

$$u = r - (JG)(\varphi \circ u), \tag{2.1}$$

where, by slight abuse of notation, $\varphi \circ u$ denotes the function $n \mapsto \varphi(n, u(n))$. Trivially, (2.1) is equivalent to

$$\Delta u = \Delta r - G(\varphi \circ u), \quad u(0) = r(0),$$

an initial-value problem associated with a nonlinear Volterra difference equation. It is clear that there exists a unique solution $u \in F(\mathbb{Z}_+, U)$ of (2.1). As is well known (see, for example, [14]), a shift-invariant operator $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$ has a *transfer function* $\mathbf{G} \in H^\infty(\mathbb{E}, \mathscr{B}(U))$ in the sense that

$$(\mathscr{Z}(Gu))(z) = \mathbf{G}(z)(\mathscr{Z}(u))(z), \quad \forall\, u \in l^2(\mathbb{Z}_+, U), \forall\, z \in \mathbb{E}.$$

We introduce the following assumption.

(A) The limit $\mathbf{G}(1) := \lim_{z \to 1, z \in \mathbb{E}} \mathbf{G}(z)$ exists and

$$\limsup_{z \to 1, z \in \mathbb{E}} \left\| \frac{1}{z-1} (\mathbf{G}(z) - \mathbf{G}(1)) \right\| < \infty.$$

2.1. A stability result of Popov type

Throughout this subsection, $U = \mathbb{R}$ and φ is time independent. The following result is a stability criterion of Popov type.

Theorem 2.1. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+))$ be a shift-invariant operator with transfer function $\mathbf{G}$ satisfying assumption* (A) *and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a non-decreasing locally integrable nonlinearity. Assume that $\mathbf{G}(1) > 0$ and there exist numbers $q \geq 0$, $\varepsilon > 0$ and $a > 0$ such that*

$$\varphi(v)v \geq \frac{1}{a}\varphi^2(v), \quad \forall\, v \in \mathbb{R}, \tag{2.2}$$

and

$$\frac{1}{a} + \operatorname{Re}\left[\left(\frac{q}{e^{i\theta}} + \frac{1}{e^{i\theta} - 1}\right)\mathbf{G}(e^{i\theta})\right] \geq \varepsilon, \quad \text{a.e. } \theta \in (0, 2\pi). \tag{2.3}$$

Let $r \in m^2(\mathbb{Z}_+)$ and let $u : \mathbb{Z}_+ \to \mathbb{R}$ be the unique solution of (2.1). Then the following statements hold.

1. *There exists a constant K (which depends only on q, ε, a and G, but not on r) such that,*

$$\|u\|_{l^\infty} + \|\nabla u\|_{l^2} + \|\varphi \circ u\|_{l^2} + (\|(\varphi \circ u)u\|_{l^1})^{1/2} + \sup_{n \geq 0} \left| \sum_{j=0}^{n} (\varphi \circ u)(j) \right| \leq K \|r\|_{m^2}. \tag{2.4}$$

2. *The limits*

$$u^\infty := \lim_{n\to\infty} u(n), \qquad \lim_{n\to\infty} \sum_{j=0}^{n} (\varphi \circ u)(j),$$

exist, are finite and, if φ *is continuous,* $\varphi(u^\infty) = 0$.

Proof. We have

$$u + (JG)(\varphi \circ u) = r, \tag{2.5}$$

or equivalently,

$$\nabla u + SG(\varphi \circ u) = \nabla r. \tag{2.6}$$

We now write (2.5) in a slightly more convenient form, namely,

$$u + H(\varphi \circ u) + \mathbf{G}(1)J(\varphi \circ u) = r, \tag{2.7}$$

where $H := JG - \mathbf{G}(1)J$. It is clear that this operator is shift-invariant with transfer function $\mathbf{H}$ given by

$$\mathbf{H}(z) = \frac{1}{z-1}[\mathbf{G}(z) - \mathbf{G}(1)], \quad \forall z \in \mathbb{E}. \tag{2.8}$$

From assumption (A) and from the fact that $\mathbf{G} \in H^\infty(\mathbb{E})$, we conclude that $\mathbf{H} \in H^\infty(\mathbb{E})$, and hence $H \in \mathscr{B}(l^2(\mathbb{Z}_+))$.

Multiplying (2.6) by q and then adding (2.7) gives

$$q(\nabla u)(j) + u(j) + G_q(\varphi \circ u)(j) + \mathbf{G}(1)(J(\varphi \circ u))(j) = q(\nabla r)(j) + r(j), \quad \forall j \in \mathbb{Z}_+, \tag{2.9}$$

where $G_q := qSG + H$. Invoking (2.8), we see that the transfer function $\mathbf{G}_q$ of G_q is given by

$$\mathbf{G}_q(z) := \frac{q}{z}\mathbf{G}(z) + \frac{1}{z-1}[\mathbf{G}(z) - \mathbf{G}(1)], \quad \forall z \in \mathbb{E}.$$

Multiplying (2.9) by $(\varphi \circ u)(j)$ and summing from 0 to n yields

$$\begin{aligned} &q\sum_{j=0}^{n} (\varphi \circ u)(j)(\nabla u)(j) + \sum_{j=0}^{n} (\varphi \circ u)(j)u(j) + \sum_{j=0}^{n} (\varphi \circ u)(j)(G_q(\varphi \circ u))(j) + \mathbf{G}(1)\sum_{j=1}^{n} (\varphi \circ u)(j)\sum_{k=0}^{j-1} (\varphi \circ u)(k) \\ &\quad = q\sum_{j=0}^{n} (\varphi \circ u)(j)(\nabla r)(j) + \sum_{j=0}^{n} (\varphi \circ u)(j)r(j). \end{aligned} \tag{2.10}$$

A simple proof by induction shows that

$$2\sum_{j=1}^{n} (\varphi \circ u)(j)\sum_{k=0}^{j-1} (\varphi \circ u)(k) = \left(\sum_{j=0}^{n} (\varphi \circ u)(j)\right)^2 - \sum_{j=0}^{n} (\varphi \circ u)^2(j). \tag{2.11}$$

Combining (2.11) with (2.10) gives

$$\begin{aligned} &q\sum_{j=0}^{n} (\varphi \circ u)(j)(\nabla u)(j) + \sum_{j=0}^{n} (\varphi \circ u)(j)u(j) + \sum_{j=0}^{n} (\varphi \circ u)(j)(G_q(\varphi \circ u))(j) \\ &\quad + \frac{\mathbf{G}(1)}{2}\left(\sum_{j=0}^{n} (\varphi \circ u)(j)\right)^2 - \frac{\mathbf{G}(1)}{2}\sum_{j=0}^{n} (\varphi \circ u)^2(j) = q\sum_{j=0}^{n} (\varphi \circ u)(j)(\nabla r)(j) + \sum_{j=0}^{n} (\varphi \circ u)(j)r(j). \end{aligned} \tag{2.12}$$

We note that

$$\operatorname{Re} \mathbf{G}_q(e^{i\theta}) = \operatorname{Re}\left[\left(\frac{q}{e^{i\theta}} + \frac{1}{e^{i\theta}-1}\right)\mathbf{G}(e^{i\theta})\right] + \frac{\mathbf{G}(1)}{2}, \quad \text{a.e. } \theta \in (0, 2\pi). \tag{2.13}$$

Combining (2.13) with (2.3), we see that

$$\operatorname{Re} \mathbf{G}_q(e^{i\theta}) = \operatorname{Re}\left[\left(\frac{q}{e^{i\theta}} + \frac{1}{e^{i\theta}-1}\right)\mathbf{G}(e^{i\theta})\right] + \frac{\mathbf{G}(1)}{2} \geq \varepsilon - \frac{1}{a} + \frac{\mathbf{G}(1)}{2}. \tag{2.14}$$

Define $v : \mathbb{Z}_+ \to \mathbb{R}$ by

$$v(j) = \begin{cases} (\varphi \circ u)(j), & \text{if } 0 \leq j \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Invoking the Parseval–Bessel identity, taking real parts and using (2.14), we derive that

$$\begin{aligned}\sum_{j=0}^{\infty} v(j)(G_q v)(j) &= \frac{1}{2\pi}\int_0^{2\pi} |\hat{v}(e^{i\theta})|^2 \mathrm{Re}\, \mathbf{G}_q(e^{i\theta})\mathrm{d}\theta \\ &\geq \frac{1}{2\pi}\left(\varepsilon - \frac{1}{a} + \frac{\mathbf{G}(1)}{2}\right)\int_0^{2\pi} |\hat{v}(e^{i\theta})|^2\,\mathrm{d}\theta \\ &= \left(\varepsilon - \frac{1}{a} + \frac{\mathbf{G}(1)}{2}\right)\sum_{j=0}^{\infty} v^2(j).\end{aligned}$$

Hence,

$$\sum_{j=0}^{n}(\varphi\circ u)(j)(G_q(\varphi\circ u))(j) \geq \left(\varepsilon - \frac{1}{a} + \frac{\mathbf{G}(1)}{2}\right)\sum_{j=0}^{n}(\varphi\circ u)^2(j). \tag{2.15}$$

Combining (2.15) and (2.12) gives

$$\begin{aligned}&q\sum_{j=0}^{n}(\varphi\circ u)(j)(\nabla u)(j) + \sum_{j=0}^{n}(\varphi\circ u)(j)u(j) + \left(\varepsilon - \frac{1}{a}\right)\sum_{j=0}^{n}(\varphi\circ u)^2(j) + \frac{\mathbf{G}(1)}{2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j)\right)^2 \\ &\leq q\sum_{j=0}^{n}(\varphi\circ u)(j)(\nabla r)(j) + \sum_{j=0}^{n}(\varphi\circ u)(j)r(j).\end{aligned} \tag{2.16}$$

Defining the function $\Phi : \mathbb{R}\to\mathbb{R}$ by $\Phi(v) := \int_0^v \varphi(\sigma)\mathrm{d}\sigma$ and using the fact that φ is non-decreasing, we estimate the first term on the LHS of (2.16) as follows:

$$\begin{aligned}\sum_{j=0}^{n}(\varphi\circ u)(j)(\nabla u)(j) &\geq \sum_{j=1}^{n}\int_{u(j-1)}^{u(j)}\varphi(\sigma)\mathrm{d}\sigma + \int_0^{u(0)}\varphi(\sigma)\mathrm{d}\sigma \\ &= \int_0^{u(n)}\varphi(\sigma)\mathrm{d}\sigma \\ &= \Phi(u(n)), \quad \forall\, n\in\mathbb{Z}_+.\end{aligned} \tag{2.17}$$

By assumption $r = r_1 + r_2\vartheta$, where $r_1 \in l^2(\mathbb{Z}_+)$ and $r_2\in\mathbb{R}$. Using this decomposition of r and invoking estimate (2.17), we obtain from (2.16) that

$$\begin{aligned}&q\Phi(u(n)) + \sum_{j=0}^{n}(\varphi\circ u)(j)u(j) + \left(\varepsilon - \frac{1}{a}\right)\sum_{j=0}^{n}(\varphi\circ u)^2(j) + \frac{\mathbf{G}(1)}{2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j)\right)^2 - \sum_{j=0}^{n}(\varphi\circ u)(j)r_2 \\ &\leq q\sum_{j=0}^{n}(\varphi\circ u)(j)(\nabla r)(j) + \sum_{j=0}^{n}(\varphi\circ u)(j)r_1(j).\end{aligned} \tag{2.18}$$

We estimate the RHS of (2.18) (using the inequality $2\alpha\beta \leq \varepsilon\alpha^2 + \beta^2/\varepsilon$ for non-negative numbers α and β) as follows:

$$\sum_{j=0}^{n}|(\varphi\circ u)(j)[q(\nabla r)(j) + r_1(j)]| \leq \frac{1}{2}\varepsilon\sum_{j=0}^{n}(\varphi\circ u)^2(j) + \frac{1}{2\varepsilon}\sum_{j=0}^{n}[q(\nabla r)(j) + r_1(j)]^2. \tag{2.19}$$

Using (2.19) in (2.18) and combining the last two terms on the LHS of (2.18) by completing the square, we arrive at

$$\begin{aligned}&q\Phi(u(n)) + \sum_{j=0}^{n}\left[(\varphi\circ u)(j)u(j) - \frac{1}{a}(\varphi\circ u)^2(j)\right] + \frac{\varepsilon}{2}\sum_{j=0}^{n}(\varphi\circ u)^2(j) + \frac{\mathbf{G}(1)}{2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j) - [\mathbf{G}(1)]^{-1}r_2\right)^2 \\ &\leq \frac{1}{2\varepsilon}\sum_{j=0}^{n}[q(\nabla r)(j) + r_1(j)]^2 + \frac{1}{2}[\mathbf{G}(1)]^{-1}{r_2}^2 \\ &\leq L\|r\|_{m^2}^2, \quad \forall\, n\in\mathbb{Z}_+,\end{aligned} \tag{2.20}$$

where L depends only on q, ε and $\mathbf{G}(1)$ and we have used the fact that $\nabla r = r_1 - Sr_1 + r_2\delta$ and so $\|\nabla r\|_{l^2} \leq 2\|r_1\|_{l^2} + |r_2|$. By sector condition (2.2) we have that

$$\sum_{j=0}^{n}\left[(\varphi\circ u)(j)u(j) - \frac{1}{a}(\varphi\circ u)^2(j)\right] \geq 0.$$

Furthermore, Φ is non-negative (since $\varphi(0) = 0$ and φ is non-decreasing) and thus, all terms on the LHS of (2.20) are non-negative. Inequality (2.20) is the key estimate from which we shall derive the theorem.

Proof of Statement 1. In the following, $K > 0$ is a generic constant which will be suitably adjusted in every step and depends only on q, ε, a and G, but not on n or r. From (2.20) we obtain

$$\frac{\varepsilon}{2}\sum_{j=0}^{n}(\varphi\circ u)^2(j) \leq L\|r\|_{m^2}^2, \quad \forall\, n \in \mathbb{Z}_+.$$

Hence,

$$\|\varphi\circ u\|_{l^2} \leq K\|r\|_{m^2}. \tag{2.21}$$

Again, by (2.20),

$$\frac{\mathbf{G}(1)}{2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j) - [\mathbf{G}(1)]^{-1}r_2\right)^2 \leq L\|r\|_{m^2}^2, \quad \forall\, n \in \mathbb{Z}_+.$$

Consequently,

$$\left|\left(\sum_{j=0}^{n}(\varphi\circ u)(j) - [\mathbf{G}(1)]^{-1}r_2\right)\right|^2 \leq K\|r\|_{m^2}^2,$$

and so

$$\left|\sum_{j=0}^{n}(\varphi\circ u)(j)\right| \leq K\|r\|_{m^2}, \quad \forall\, n \in \mathbb{Z}_+.$$

Hence,

$$\sup_{n\geq 0}\left|\sum_{j=0}^{n}(\varphi\circ u)(j)\right| \leq K\|r\|_{m^2}. \tag{2.22}$$

Again starting with (2.20), we obtain the inequality

$$\sum_{j=0}^{n}\left[(\varphi\circ u)(j)u(j) - \frac{1}{a}(\varphi\circ u)^2(j)\right] \leq L\|r\|_{m^2}^2, \quad \forall\, n \in \mathbb{Z}_+.$$

It follows that

$$0 \leq \sum_{j=0}^{n}(\varphi\circ u)(j)u(j) \leq L\|r\|_{m^2}^2 + \frac{1}{a}\sum_{j=0}^{\infty}(\varphi\circ u)^2(j) \leq K\|r\|_{m^2}^2, \quad \forall\, n \in \mathbb{Z}_+,$$

where we have used (2.21). Consequently,

$$(\|(\varphi\circ u)u\|_{l^1})^{1/2} \leq K\|r\|_{m^2}. \tag{2.23}$$

Using the boundedness of H together with (2.21) gives

$$|(H(\varphi\circ u))(j)| \leq \|H(\varphi\circ u)\|_{l^2} \leq K\|r\|_{m^2}. \tag{2.24}$$

Invoking (2.7), we arrive at

$$\begin{aligned}|u(j)| &\leq |(H(\varphi\circ u))(j)| + |\mathbf{G}(1)||(J(\varphi\circ u))(j)| + |r(j)|\\ &\leq |(H(\varphi\circ u))(j)| + |\mathbf{G}(1)||(J(\varphi\circ u))(j)| + \|r\|_{m^2}.\end{aligned} \tag{2.25}$$

Taking the supremum and appealing to (2.22) and (2.24), we obtain from (2.25) that

$$\|u\|_{l^\infty} = \sup_{n\geq 0}|u(n)| \leq K\|r\|_{m^2}. \tag{2.26}$$

By (2.6), we have that

$$\nabla u = \nabla r - SG(\varphi \circ u). \tag{2.27}$$

Since $\varphi \circ u \in l^2(\mathbb{Z}_+)$ by (2.21), $G \in \mathscr{B}(l^2(\mathbb{Z}_+))$ and $\nabla r \in l^2(\mathbb{Z}_+)$ we conclude that $\nabla u \in l^2(\mathbb{Z}_+)$. Furthermore, by (2.21) the l^2-norm of the RHS of (2.27) is bounded by $K\|r\|_{m^2}$, and thus $\|\nabla u\|_{l^2} \leq K\|r\|_{m^2}$. Combining this with (2.21), (2.22), (2.23) and (2.26), it is clear that (2.4) holds.

Proof of Statement 2. We note that, by (2.12),

$$\frac{\mathbf{G}(1)}{2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j)\right)^2 - \sum_{j=0}^{n}(\varphi\circ u)(j)r_2 = \frac{\mathbf{G}(1)}{2}\sum_{j=0}^{n}(\varphi\circ u)^2(j) - q\sum_{j=0}^{n}(\varphi\circ u)(j)(\nabla u)(j)$$
$$-\sum_{j=0}^{n}(\varphi\circ u)(j)u(j) - \sum_{j=0}^{n}(\varphi\circ u)(j)(G_q(\varphi\circ u))(j) + q\sum_{j=0}^{n}(\varphi\circ u)(j)(\nabla r)(j) + \sum_{j=0}^{n}(\varphi\circ u)(j)r_1(j). \tag{2.28}$$

Completing the square on the LHS of (2.28), we obtain

$$\frac{\mathbf{G}(1)}{2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j) - [\mathbf{G}(1)]^{-1}r_2\right)^2 = \frac{1}{2}[\mathbf{G}(1)]^{-1}r_2{}^2 - \sum_{j=0}^{n}(\varphi\circ u)(j)u(j) - \sum_{j=0}^{n}(\varphi\circ u)(j)(G_q(\varphi\circ u))(j)$$
$$+\sum_{j=0}^{n}(\varphi\circ u)(j)r_1(j) + \frac{\mathbf{G}(1)}{2}\sum_{j=0}^{n}(\varphi\circ u)^2(j)$$
$$+q\sum_{j=0}^{n}(\varphi\circ u)(j)(\nabla r)(j) - q\sum_{j=0}^{n}(\varphi\circ u)(j)(\nabla u)(j). \tag{2.29}$$

By statement 1, $\varphi\circ u \in l^2(\mathbb{Z}_+)$, $\nabla u \in l^2(\mathbb{Z}_+)$ and $(\varphi\circ u)u \in l^1(\mathbb{Z}_+)$. Since $r_1 \in l^2(\mathbb{Z}_+)$ and $G_q \in \mathscr{B}(l^2(\mathbb{Z}_+))$, it follows that $(\varphi\circ u)G_q(\varphi\circ u)$, $(\varphi\circ u)r_1$, $(\varphi\circ u)^2$, $(\varphi\circ u)(\nabla r)$ and $(\varphi\circ u)(\nabla u)$ are in $l^1(\mathbb{Z}_+)$, so that the RHS of (2.29) has a finite limit as $n\to\infty$. Hence,

$$\lambda := \lim_{n\to\infty}\left(\sum_{j=0}^{n}(\varphi\circ u)(j) - [\mathbf{G}(1)]^{-1}r_2\right)^2$$

exists and is finite. Set $w(n) := \sum_{j=0}^{n}(\varphi\circ u)(j)$. By statement 1, $\varphi\circ u \in l^2(\mathbb{Z}_+)$, so that, in particular, $(\varphi\circ u)(j)\to 0$ as $j\to\infty$. Hence, $(\Delta w)(n) = (\varphi\circ u)(n+1)\to 0$ and we see that $\sum_{j=0}^{n}(\varphi\circ u)(j)$ converges to one of the points $[\mathbf{G}(1)]^{-1}r_2 \pm \sqrt{\lambda}$. To prove that $\lim_{n\to\infty}u(n)$ exists, it is sufficient to show that

$$\lim_{n\to\infty}(u(n) + \mathbf{G}(1)(J(\varphi\circ u))(n)) = r_2. \tag{2.30}$$

By (2.7), Eq. (2.30) is equivalent to the claim that $\lim_{n\to\infty}(H(\varphi\circ u))(n) = 0$. The later follows from the fact that, $H \in \mathscr{B}(l^2(\mathbb{Z}_+))$ and $\varphi\circ u \in l^2(\mathbb{Z}_+)$. Therefore, $u^\infty := \lim_{n\to\infty}u(n)$ exists and is finite. Finally, since by statement 1, $\varphi\circ u \in l^2(\mathbb{Z}_+)$, it is clear that, if φ is continuous, then $\varphi(u^\infty) = 0$. □

In a sense, Theorem 2.1 forms the discrete-time counterpart of a continuous-time stability result proved in [6] (see Theorem 3.1 in [6]). In the literature on discrete-time systems, the result closest to Theorem 2.1 is part (b) of Theorem 5 on p. 192 in [1]. However, there it is assumed that the linear system is l^2-stable, whilst in the present paper the linear system is given by GJ, which is not l^p-stable for any $p \geq 1$.

2.2. A stability result of circle-criterion type

In this subsection, U is an arbitrary real or complex (possibly infinite-dimensional) Hilbert space. We consider the system shown in Fig. 2 with time-varying φ. The following theorem is a stability result of circle-criterion type.

Theorem 2.2. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$ be a shift-invariant operator with transfer function $\mathbf{G}$ satisfying assumption* (A) *with $\mathbf{G}(1)$ invertible, and let $\varphi : \mathbb{Z}_+ \times U \to U$ be a time-varying nonlinearity. Assume that there exist self-adjoint $P \in \mathscr{B}(U)$, invertible $Q \in \mathscr{B}(U)$ with $Q\mathbf{G}(1) = [Q\mathbf{G}(1)]^* \geq 0$ and a number $\varepsilon > 0$ such that*

$$\operatorname{Re}\langle\varphi(n,v), Qv\rangle \geq \langle\varphi(n,v), P\varphi(n,v)\rangle, \quad \forall\, n\in\mathbb{Z}_+, v\in U, \tag{2.31}$$

and

$$P + \frac{1}{2}\left[\frac{1}{e^{i\theta}-1}Q\mathbf{G}(e^{i\theta}) + \frac{1}{e^{-i\theta}-1}\mathbf{G}^*(e^{i\theta})Q^*\right] \geq \varepsilon I, \quad \text{a.e. } \theta\in(0,2\pi). \tag{2.32}$$

Let $r \in m^2(\mathbb{Z}_+, U)$, that is, $r = r_1 + r_2\vartheta$ with $r_1 \in l^2(\mathbb{Z}_+, U)$ and $r_2 \in U$, and let $u : \mathbb{Z}_+ \to U$ be the unique solution of (2.1). Then the following statements hold.

1. *There exists a constant K (which depends only on ε, P, Q and G, but not on r) such that,*
$$\|u\|_{l^\infty} + \|\nabla u\|_{l^2} + +\|\varphi \circ u\|_{l^2} + (\|\mathrm{Re}\,\langle(\varphi \circ u), Qu\rangle\|_{l^1})^{1/2} + \sup_{n\geq 0}\left\|\sum_{j=0}^{n}(\varphi \circ u)(j)\right\| \leq K\|r\|_{m^2}. \tag{2.33}$$
2. *We have,*
$$\lim_{n\to\infty}\left(u(n) + \mathbf{G}(1)\sum_{j=0}^{n}(\varphi \circ u)(j)\right) = r_2; \tag{2.34}$$
in particular $\lim_{n\to\infty} u(n)$ exists if and only if $\lim_{n\to\infty}\sum_{j=0}^{n}(\varphi \circ u)(j)$ exists, in which case
$$\lim_{n\to\infty} u(n) = r_2 - \mathbf{G}(1)\lim_{n\to\infty}\sum_{j=0}^{n}(\varphi \circ u)(j). \tag{2.35}$$
3. *There exists a sphere $S_0 \subset U$ centred at 0, such that*
$$\lim_{n\to\infty}\mathrm{dist}(u(n), \mathbf{G}(1)[Q\mathbf{G}(1)]^{-1/2}S_0) = 0, \tag{2.36}$$
in particular, if $\dim U = 1$, then $\lim_{n\to\infty} u(n)$ and $\lim_{n\to\infty}\sum_{j=0}^{n}(\varphi \circ u)(j)$ exist.
4. *If we relax condition (2.31) and only require that for some $n_0 > 0$,*
$$\mathrm{Re}\,\langle\varphi(n, v), Qv\rangle \geq \langle\varphi(n, v), P\varphi(n, v)\rangle, \quad \forall\, n \geq n_0, \forall\, v \in U, \tag{2.37}$$
then the LHS of (2.33) is still finite (but no longer bounded in terms of $\|r\|_{m^2}$) and statements 2 and 3 remain valid.
5. *Under the additional assumptions*
(**B**) *φ does not depend on time,*
(**C**) *$\varphi^{-1}(0) \cap B$ is precompact for every bounded set $B \subset U$,*
(**D**) *$\inf_{v\in B}\|\varphi(v)\| > 0$ for every bounded, closed, non-empty set $B \subset U$ such that $\varphi^{-1}(0) \cap B = \emptyset$,*
we have that $\lim_{n\to\infty}\mathrm{dist}(u(n), \varphi^{-1}(0)) = 0$.
6. *If the additional assumptions (**B**)–(**D**) of statement 5 hold and if the intersection $\mathrm{cl}(\varphi^{-1}(0)) \cap \mathbf{G}(1)[Q\mathbf{G}(1)]^{-1/2}S_0$ is totally disconnected for every sphere $S_0 \subset U$ centred at 0, then $u(n)$ converges as $n \to \infty$.*

Remark 2.3. (i) If $\dim U < \infty$ and assumption (B) holds, then assumption (D) is satisfied. Moreover, if $\dim U < \infty$, assumption (B) holds and φ is continuous, then assumption (C) is satisfied.

(ii) Theorem 2.2 forms the discrete-time counterpart of a continuous-time stability result proved in [6] (see Theorem 3.3 in [6]). In the literature on discrete-time systems, the result closest to Theorem 2.2 is case (2) of Theorem 37 on p. 370 in [3]. However, there it is assumed that the impulse response of the linear system is in l^1 (implying that the linear system is l^p-stable for every $p \geq 1$): this assumption is not satisfied for the linear system with input–output operator given by GJ.

Before proving Theorem 2.2, we state a slightly simplified version of this result (where P and Q are scalars and $P \geq 0$) in the form of a corollary which is convenient in the context of applications of Theorem 2.2 to integral control (see Section 3).

Corollary 2.4. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$ be a shift-invariant operator with transfer function $\mathbf{G}$ satisfying assumption* (A) *with $\mathbf{G}(1)$ invertible and $\mathbf{G}(1) = \mathbf{G}^*(1) \geq 0$, and let $\varphi : \mathbb{Z}_+ \times U \to U$ be a time-varying nonlinearity. Assume that there exist numbers $\varepsilon > 0$ and $a > 0$ such that*
$$\mathrm{Re}\,\langle\varphi(n, v), v\rangle \geq \frac{1}{a}\|\varphi(n, v)\|^2, \quad \forall\, v \in U, \tag{2.38}$$
and
$$\frac{1}{a}I + \frac{1}{2}\left[\frac{1}{e^{i\theta} - 1}\mathbf{G}(e^{i\theta}) + \frac{1}{e^{-i\theta} - 1}\mathbf{G}^*(e^{i\theta})\right] \geq \varepsilon I, \quad \text{a.e. } \theta \in (0, 2\pi). \tag{2.39}$$
Then, for all $r \in m^2(\mathbb{Z}_+, U)$, the conclusions of Theorem 2.2 hold with $P = (1/a)I$ and $Q = I$.

Proof of Theorem 2.2. Since u satisfies (2.1), we conclude that
$$Qu(j) + (G_Q(\varphi \circ u))(j) + Q\mathbf{G}(1)(J(\varphi \circ u))(j) = Qr(j), \quad \forall\, j \in \mathbb{Z}_+, \tag{2.40}$$
where the operator $G_Q := Q(JG - \mathbf{G}(1)J)$ is shift-invariant with transfer function $\mathbf{G}_Q$ given by
$$\mathbf{G}_Q(z) := \frac{1}{z - 1}Q[\mathbf{G}(z) - \mathbf{G}(1)], \quad \forall\, z \in \mathbb{E}.$$

From assumption (A) and from the fact that $\mathbf{G} \in H^\infty(\mathbb{E}, \mathscr{B}(U))$, we conclude that $\mathbf{G}_Q \in H^\infty(\mathbb{E}, \mathscr{B}(U))$, and hence $G_Q \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$.

Forming the inner product with $(\varphi \circ u)(j)$, taking real parts and summing from 0 to n in (2.40) yields

$$\begin{aligned}&\operatorname{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), Qu(j) \rangle + \operatorname{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), (G_Q(\varphi \circ u))(j) \rangle \\ &+ \operatorname{Re} \sum_{j=1}^{n} \left\langle (\varphi \circ u)(j), Q\mathbf{G}(1) \sum_{k=0}^{j-1} (\varphi \circ u)(k) \right\rangle = \operatorname{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), Qr(j) \rangle. \end{aligned} \tag{2.41}$$

Since, by assumption, $Q\mathbf{G}(1) = [Q\mathbf{G}(1)]^* \geq 0$, the square root $[Q\mathbf{G}(1)]^{1/2}$ of $Q\mathbf{G}(1)$ exists. A simple proof by induction then yields the following identity

$$\operatorname{Re} \sum_{j=1}^{n} \left\langle (\varphi \circ u)(j), Q\mathbf{G}(1) \sum_{k=0}^{j-1} (\varphi \circ u)(k) \right\rangle = \frac{1}{2} \left\| [Q\mathbf{G}(1)]^{1/2} \sum_{j=0}^{n} (\varphi \circ u)(j) \right\|^2 - \frac{1}{2} \sum_{j=0}^{n} \| [Q\mathbf{G}(1)]^{1/2} (\varphi \circ u)(j) \|^2. \tag{2.42}$$

Combining (2.41) with (2.42) gives

$$\begin{aligned}&\operatorname{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), Qu(j) \rangle + \operatorname{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), (G_Q(\varphi \circ u))(j) \rangle \\ &+ \frac{1}{2} \left\| [Q\mathbf{G}(1)]^{1/2} \sum_{j=0}^{n} (\varphi \circ u)(j) \right\|^2 - \frac{1}{2} \sum_{j=0}^{n} \| [Q\mathbf{G}(1)]^{1/2} (\varphi \circ u)(j) \|^2 \\ &= \operatorname{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), Qr(j) \rangle. \end{aligned} \tag{2.43}$$

Now an argument very similar to that in the proof of Theorem 2.1 can be adopted: invoking the Parseval–Bessel identity and the frequency-domain condition (2.32), it can be shown that

$$\begin{aligned}&\operatorname{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), (G_Q(\varphi \circ u))(j) \rangle \\ &\geq \varepsilon \sum_{j=0}^{n} \| (\varphi \circ u)(j) \|^2 - \sum_{j=0}^{n} \langle (\varphi \circ u)(j), P(\varphi \circ u)(j) \rangle + \frac{1}{2} \sum_{j=0}^{n} \| [Q\mathbf{G}(1)]^{1/2} (\varphi \circ u)(j) \|^2, \end{aligned} \tag{2.44}$$

which together with (2.43) can be used to derive the key inequality

$$\begin{aligned}&\operatorname{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), Qu(j) - P(\varphi \circ u)(j) \rangle + \frac{\varepsilon}{2} \sum_{j=0}^{n} \| (\varphi \circ u)(j) \|^2 + \frac{1}{2} \left\| [Q\mathbf{G}(1)]^{1/2} \left(\sum_{j=0}^{n} (\varphi \circ u)(j) - [\mathbf{G}(1)]^{-1} r_2 \right) \right\|^2 \\ &\leq \frac{1}{2\varepsilon} \sum_{j=0}^{\infty} \| Qr_1(j) \|^2 + \frac{1}{2} \| [Q\mathbf{G}(1)]^{1/2} [\mathbf{G}(1)]^{-1} r_2 \|^2 \\ &\leq L \| r \|_{m^2}^2, \quad \forall\, n \in \mathbb{Z}_+, \end{aligned} \tag{2.45}$$

where L depends only on ε, G and Q. By the sector condition (2.31),

$$\operatorname{Re} \langle (\varphi \circ u)(j), Qu(j) - P(\varphi \circ u)(j) \rangle \geq 0, \quad \forall\, j \in \mathbb{Z}_+,$$

showing that all terms on the LHS of (2.45) are non-negative.

Proof of Statement 1. In the following, $K > 0$ is a generic constant which will be suitably adjusted in every step and depends only on ε, P, Q and G, but not on n or r. Invoking (2.45), it now follows easily (see proof of Theorem 2.1) that

$$\| u \|_{l^\infty} + \| \nabla u \|_{l^2} + \| \varphi \circ u \|_{l^2} + \sup_{n \geq 0} \left\| \sum_{j=0}^{n} (\varphi \circ u)(j) \right\| \leq K \| r \|_{m^2}. \tag{2.46}$$

Furthermore, by (2.31) and (2.45),

$$0 \leq \operatorname{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), Qu(j) - P(\varphi \circ u)(j) \rangle \leq L \| r \|_{m^2}^2, \quad \forall\, n \in \mathbb{Z}_+,$$

Since,

$$|\mathrm{Re}\,\langle(\varphi\circ u)(j), Qu(j)\rangle| \le |\mathrm{Re}\,\langle(\varphi\circ u)(j), Qu(j) - P(\varphi\circ u)(j)\rangle| + |\mathrm{Re}\,\langle(\varphi\circ u)(j), P(\varphi\circ u)(j)\rangle|$$

it follows that,

$$\sum_{j=0}^{n}|\mathrm{Re}\,\langle(\varphi\circ u)(j), Qu(j)\rangle| \le L\|r\|_{m^2}^2 + \|P\|\sum_{j=0}^{\infty}\|(\varphi\circ u)^2(j)\| \le K\|r\|_{m^2}^2, \quad \forall\, n\in\mathbb{Z}_+,$$

where we have used (2.46). Consequently,

$$(\|\mathrm{Re}\,\langle(\varphi\circ u), Qu\rangle\|_{l^1})^{1/2} \le K\|r\|_{m^2},$$

which combined with (2.46) shows that (2.33) holds.

Proof of Statement 2. Since $(\varphi\circ u)(n)\to 0$ as $n\to\infty$ (by statement 1) and Q is invertible, it follows from (2.40) that (2.34) is equivalent to the claim that

$$\lim_{n\to\infty}(G_Q(\varphi\circ u))(n) = 0.$$

Since $G_Q\in\mathscr{B}(l^2(\mathbb{Z}_+,U))$ and $\varphi\circ u\in l^2(\mathbb{Z}_+,U)$, we conclude that $G_Q(\varphi\circ u)(n)\to 0$ as $n\to\infty$. It is now clear that if $\lim_{n\to\infty}u(n)$ exists, then, by (2.34), $\lim_{n\to\infty}\sum_{j=0}^{n}(\varphi\circ u)(j)$ exists and (2.35) follows trivially.

Proof of Statement 3. We re-write (2.43) as follows:

$$\begin{aligned}&\frac{1}{2}\left\|[Q\mathbf{G}(1)]^{1/2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j) - [\mathbf{G}(1)]^{-1}r_2\right)\right\|^2\\ &\quad= \frac{1}{2}\|[Q\mathbf{G}(1)]^{1/2}[\mathbf{G}(1)]^{-1}r_2\|^2 - \mathrm{Re}\sum_{j=0}^{n}\langle(\varphi\circ u)(j), Qu(j)\rangle + \mathrm{Re}\sum_{j=0}^{n}\langle(\varphi\circ u)(j), Qr_1(j)\rangle\\ &\qquad- \mathrm{Re}\sum_{j=0}^{n}\langle(\varphi\circ u)(j), (G_Q(\varphi\circ u))(j)\rangle + \frac{1}{2}\sum_{j=0}^{n}\|[Q\mathbf{G}(1)]^{1/2}(\varphi\circ u)(j)\|^2.\end{aligned}\tag{2.47}$$

The RHS of (2.47) has a finite limit as $n\to\infty$ since $\mathrm{Re}\,\langle(\varphi\circ u), Qu\rangle$, $\langle(\varphi\circ u), Qr_1\rangle$, $\langle(\varphi\circ u), G_Q(\varphi\circ u)\rangle$ and $\|(\varphi\circ u)\|^2$ are in $l^1(\mathbb{Z}_+,\mathbb{C})$. Hence we have that,

$$\lim_{n\to\infty}\left\|[Q\mathbf{G}(1)]^{1/2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j) - [\mathbf{G}(1)]^{-1}r_2\right)\right\|$$

exists. Set

$$\lambda := \lim_{n\to\infty}\left\|[Q\mathbf{G}(1)]^{1/2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j) - [\mathbf{G}(1)]^{-1}r_2\right)\right\|.\tag{2.48}$$

Now using (2.34) and writing $f(n) := \sum_{j=0}^{n}(\varphi\circ u)(j)$ we have,

$$\lim_{n\to\infty}(u(n) - r_2 + \mathbf{G}(1)f(n)) = 0.\tag{2.49}$$

Let $g(n) := [Q\mathbf{G}(1)]^{1/2}(f(n) - [\mathbf{G}(1)]^{-1}r_2)$. Then from (2.48) we have $\lim_{n\to\infty}\|g(n)\| = \lambda$, that is, $g(n)$ approaches the sphere S_0 of radius λ centred at 0 as $n\to\infty$. So by (2.49),

$$0 = \lim_{n\to\infty}(u(n) - r_2 + \mathbf{G}(1)f(n)) = \lim_{n\to\infty}(u(n) + \mathbf{G}(1)[Q\mathbf{G}(1)]^{-1/2}g(n))$$

and we see that $u(n)$ approaches the set $\mathbf{G}(1)[Q\mathbf{G}(1)]^{-1/2}S_0$. Finally if $\dim U = 1$, then the sphere S_0 consists of just one or two points. To see that the sequence $u(n)$ does not oscillate between small neighbourhoods of each of these two points, we observe that $(\nabla u)(n)\to 0$ as $n\to\infty$ (since $\nabla u\in l^2(\mathbb{Z}_+,U)$ by statement 1). Therefore, $\lim_{n\to\infty}u(n)$ exists. It is now clear from statement 2 that $\lim_{n\to\infty}\sum_{j=0}^{n}(\varphi\circ u)(j)$ exists.

Proof of Statement 4. In proving statement 4, we use (2.45), the only problem being that the sector condition now only holds whenever $j\ge n_0$. Hence the first term on the LHS of (2.45) could have either sign. However, setting

$$\Gamma := \mathrm{Re}\sum_{j=0}^{n_0}|\langle(\varphi\circ u)(j), Qu(j) - P(\varphi\circ u)(j)\rangle|$$

we see from (2.45) that for all $n \geq n_0$,

$$\operatorname{Re} \sum_{j=n_0}^{n} \langle (\varphi \circ u)(j), Qu(j) - P(\varphi \circ u)(j) \rangle + \frac{\varepsilon}{2} \sum_{j=0}^{n} \|(\varphi \circ u)(j)\|^2 + \frac{1}{2} \left\| [Q\mathbf{G}(1)]^{1/2} \left(\sum_{j=0}^{n} (\varphi \circ u)(j) - [\mathbf{G}(1)]^{-1} r_2 \right) \right\|^2$$

$$\leq \Gamma + \frac{1}{2\varepsilon} \sum_{j=0}^{n} \|Qr_1(j)\|^2 + \frac{1}{2} \|[Q\mathbf{G}(1)]^{1/2} [\mathbf{G}(1)]^{-1} r_2\|^2. \tag{2.50}$$

Now by (2.37), the first term on the LHS of (2.50) is non-negative, so that all terms on the LHS of (2.50) are non-negative. The same arguments invoked in the proof of statements 1–3 can now be used to complete the proof of statement 4.

Proof of Statement 5. Assume that the additional assumptions (B), (C) and (D) are satisfied. Since u is bounded, there exists a closed bounded set $B \subset U$ such that $u(n) \in B$ for all $n \in \mathbb{Z}_+$. It follows from (C) that $\varphi^{-1}(0) \cap B$ is precompact. Consequently, for given $\eta > 0$, $\varphi^{-1}(0) \cap B$ is contained in a finite union of open balls with radius η, each ball centred at some point in $\operatorname{cl}(\varphi^{-1}(0) \cap B)$. Denoting this union by B_η, we claim that $u(n) \in B_\eta$ for all sufficiently large n. This is trivially true if $B \subset B_\eta$. If not, then the set $C := B \setminus B_\eta$ is non-empty. Moreover, C is bounded and closed with $\varphi^{-1}(0) \cap C = \emptyset$ and so $\inf_{v \in C} \|\varphi(v)\| > 0$ by (D). We know from statement 1 that $\varphi \circ u \in l^2(\mathbb{Z}_+, U)$, hence $\lim_{n\to\infty} (\varphi \circ u)(n) = 0$, and so also in this case $u(n) \in B_\eta$ for all sufficiently large n. This implies that

$$\lim_{n\to\infty} \operatorname{dist}(u(n), \varphi^{-1}(0) \cap B) = 0 \tag{2.51}$$

and, a fortiori,

$$\lim_{n\to\infty} \operatorname{dist}(u(n), \varphi^{-1}(0)) = 0, \tag{2.52}$$

completing the proof of statement 5.

Proof of Statement 6. To verify statement 6, we note that, by (2.51) and precompactness of $\varphi^{-1}(0) \cap B$, the set $\{u(n) : n \in \mathbb{Z}_+\}$ is precompact. Consequently, the ω-limit set Ω of u is non-empty, compact and is approached by $u(n)$ as $n \to \infty$. Invoking (2.52), we see that $\Omega \subset \operatorname{cl}(\varphi^{-1}(0))$. Furthermore, by statement 3, we know that $u(n)$ approaches $\mathbf{G}(1)[Q\mathbf{G}(1)]^{-1/2} S_0$ for some sphere $S_0 \subset U$ centred at 0, hence $\Omega \subset \mathbf{G}(1)[Q\mathbf{G}(1)]^{-1/2} S_0$. Hence,

$$\Omega \subset \operatorname{cl}(\varphi^{-1}(0)) \cap \mathbf{G}(1)[Q\mathbf{G}(1)]^{-1/2} S_0. \tag{2.53}$$

By statement 1, $(\nabla u)(n) \to 0$ as $n \to \infty$. Consequently, by a standard result, it follows that Ω is connected. On the other hand, by hypothesis, $\operatorname{cl}(\varphi^{-1}(0)) \cap \mathbf{G}(1)[Q\mathbf{G}(1)]^{-1/2} S_0$ is totally disconnected, implying via (2.53) that Ω is totally disconnected. As a consequence, Ω must consist of exactly one point, or, equivalently, $u(n)$ converges as $n \to \infty$. □

Remark 2.5. In addition to the strictly causal operator J, there is another "natural" discrete-time integrator, namely $J_I := J + I$, that is,

$$(J_I v)(n) = \sum_{j=0}^{n} v(j), \quad \forall\, n \in \mathbb{Z}_+, \forall\, v \in F(\mathbb{Z}_+, U), \tag{2.54}$$

which is an integrator with feedthrough operator equal to I. Obviously, if G is not strictly causal, then the equation

$$u = r - (J_I G)(\varphi \circ u), \tag{2.55}$$

may not have any solution or may have multiple solutions. Trivially, if, for every $n \in \mathbb{Z}_+$, the map

$$U \to U, \quad \xi \mapsto \xi + \mathbf{G}(\infty)\varphi(n, \xi),$$

where $\mathbf{G}(\infty) := \lim_{z\to\infty} \mathbf{G}(z)$, is surjective (bijective, respectively), then (2.55) has at least one solution (a unique solution, respectively). Furthermore, it is easy to see that, if we replace (2.1) by (2.55) and (2.3) by

$$\frac{1}{a} + \operatorname{Re}\left[\left(q + \frac{e^{i\theta}}{e^{i\theta} - 1} \right) \mathbf{G}(e^{i\theta}) \right] \geq \varepsilon, \quad \text{a.e. } \theta \in (0, 2\pi), \tag{2.56}$$

and (2.32) by,

$$P + \frac{1}{2} \left[\frac{e^{i\theta}}{e^{i\theta} - 1} Q\mathbf{G}(e^{i\theta}) + \frac{e^{-i\theta}}{e^{-i\theta} - 1} \mathbf{G}^*(e^{i\theta}) Q^* \right] \geq \varepsilon I, \quad \text{a.e. } \theta \in (0, 2\pi),$$

then the conclusions of Theorems 2.1 and 2.2 remain valid for every solution of (2.55) and furthermore, in the case of Theorem 2.1, the range of values for a can be extended to include $a = \infty$ (this allows the inclusion of certain cubic nonlinearities).

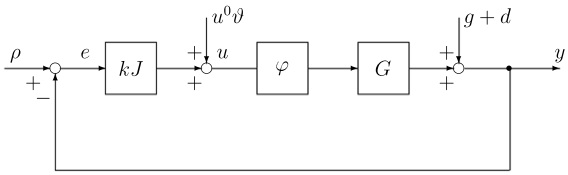

Fig. 3. Integral control in the presence of an input nonlinearity.

3. Application to integral control in the presence of input/output nonlinearities

In this section we apply Theorem 2.1 and Corollary 2.4 to derive results on low-gain integral control in the presence of input/output nonlinearities and output disturbances. Throughout this section it is assumed that $U = \mathbb{R}$.

3.1. Integral control in the presence of input nonlinearities

Consider the feedback system shown in Fig. 3, where $\rho \in \mathbb{R}$ is a constant, $k \in \mathbb{R}$ is a gain parameter, $u^0 \in \mathbb{R}$ is the initial state of the integrator (or, equivalently, the initial value of u), $\varphi : \mathbb{R} \to \mathbb{R}$ is a static input nonlinearity and $G \in \mathscr{B}(l^2(\mathbb{Z}_+))$ is a shift-invariant operator with transfer function denoted by $\mathbf{G}$. The function g models the effect of non-zero initial conditions of the system with input–output operator G and the function d is an external disturbance.

The feedback system shown in Fig. 3 is described by the following equation

$$u = u^0\vartheta + kJ(\rho\vartheta - g - d - G(\varphi \circ u)). \tag{3.1}$$

Trivially, (3.1) can be written in the form

$$(\Delta u)(n) = k[\rho - (g(n) + d(n) + (G(\varphi \circ u))(n))], \quad n \in \mathbb{Z}_+, \quad u(0) = u^0 \in \mathbb{R},$$

an initial-value problem associated with a nonlinear Volterra difference equation. It is easy to show that (3.1) has a unique solution $u \in F(\mathbb{Z}_+)$.

The aim in this subsection is to choose the gain parameter k such that the tracking error,

$$e(n) := \rho - (g(n) + d(n) + (G(\varphi \circ u))(n)) = (\Delta u)(n)/k$$

converges to 0 as $n \to \infty$.

In Theorem 3.2, the main result of this subsection, we shall impose the following assumption on $\mathbf{G}$.

($\mathbf{A}'$) The limits

$$\mathbf{G}(1) := \lim_{z\to 1, z\in\mathbb{E}} \mathbf{G}(z) \quad \text{and} \quad \mathbf{G}'(1) := \lim_{z\to 1, z\in\mathbb{E}} \frac{\mathbf{G}(z) - \mathbf{G}(1)}{z-1}$$

exist, and, furthermore,

$$\limsup_{z\to 1, z\in\mathbb{E}} \left| \frac{\mathbf{G}(z) - \mathbf{G}(1) - (z-1)\mathbf{G}'(1)}{(z-1)^2} \right| < \infty.$$

Remark 3.1. (i) Note that if $\mathbf{G}$ satisfies assumption (A′), then $\mathbf{G}$ satisfies assumption (A).

(ii) If $\mathbf{G}$ extends analytically into a neighbourhood of 1, then (A′) holds.

We define

$$f(G) := \sup_{q\geq 0} \left\{ \operatorname{essinf}_{\theta\in(0,2\pi)} \operatorname{Re}\left[\left(\frac{q}{e^{i\theta}} + \frac{1}{e^{i\theta}-1} \right) \mathbf{G}(e^{i\theta}) \right] \right\}.$$

If the transfer function $\mathbf{G}$ of G satisfies assumption (A), then it can be shown that $-\infty < f(G) \leq -\mathbf{G}(1)/2$, see [15].

For $a \in (0, \infty)$, we let $\mathscr{S}(a)$ denote the set of all functions $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying the sector condition

$$0 \leq \varphi(v)v \leq av^2, \quad \forall\, v \in \mathbb{R}.$$

Trivially, $\varphi \in \mathscr{S}(a)$ if and only if $\varphi(v)v \geq \varphi^2(v)/a$ for all $v \in \mathbb{R}$.

Theorem 3.2. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+))$ be a shift-invariant operator with transfer function $\mathbf{G}$. Assume that assumption (A′) holds with $\mathbf{G}(1) > 0$, that $Jg \in m^2(\mathbb{Z}_+)$, $d = d_1 + d_2\vartheta$ with $Jd_1 \in m^2(\mathbb{Z}_+)$ and $d_2 \in \mathbb{R}$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous and non-decreasing. Let $\rho \in \mathbb{R}$, assume that $(\rho - d_2)/\mathbf{G}(1) \in \operatorname{im}\varphi$ and let u be the solution of (3.1). Under these conditions the following statements hold.*

1. *If $\varphi - \varphi(0) \in \mathscr{S}(a)$ for some $a \in (0, \infty)$, then there exists a constant $k^* \in (0, \infty)$ (depending on G, φ and ρ) such that for all $k \in (0, k^*)$, the limit $\lim_{n\to\infty} u(n) =: u^\infty$ exists and satisfies $\varphi(u^\infty) = (\rho - d_2)/\mathbf{G}(1)$,*

$$e = \Delta u/k \in l^2(\mathbb{Z}_+) \quad \text{and} \quad \varphi \circ u - \varphi(u^\infty)\vartheta \in l^2(\mathbb{Z}_+),$$

so that, in particular, $\lim_{n\to\infty} e(n) = 0$.

2. *If φ is globally Lipschitz continuous with Lipschitz constant $\lambda > 0$, then the conclusions of statement 1 are valid with $k^* = 1/|\lambda f(G)|$.*

Remark 3.3. (i) Note that in statement 2 of Theorem 3.2, the constant k^* depends only on G and the Lipschitz constant of φ, but not on ρ.

(ii) It can be shown that g and d_1 satisfy the assumptions in Theorem 3.2 if $(Jg)(n)$, $(Jd_1)(n)$ converge to finite limits as $n \to \infty$ and the functions $n \mapsto \sum_{k=n}^{\infty} g(k)$, $n \mapsto \sum_{k=n}^{\infty} d_1(k)$ are in $l^2(\mathbb{Z}_+)$, see [15] for details.

(iii) Another sufficient condition for g and d_1 to satisfy the assumptions in Theorem 3.2 is that the functions $j \mapsto g(j)j^\alpha$ and $j \mapsto d_1(j)j^\alpha$ are in $l^2(\mathbb{Z}_+)$ for some $\alpha > 1$, see [15] for details.

(iv) In general d_2 is unknown, but it is reasonable to assume that $d_2 \in [\alpha, \beta]$, where α and β are known constants. The condition

$$(\rho - \alpha)/\mathbf{G}(1), (\rho - \beta)/\mathbf{G}(1) \in \mathrm{im}\varphi$$

does not involve d_2 and is sufficient for $(\rho - d_2)/\mathbf{G}(1)$ to be in $\mathrm{im}\varphi$. Furthermore, if φ is continuous and $\lim_{\xi \to \pm\infty} \varphi(\xi) = \pm\infty$, then the assumption $(\rho - d_2)/\mathbf{G}(1) \in \mathrm{im}\varphi$ is automatically satisfied.

Proof of Theorem 3.2. Choose $u^\rho \in \mathbb{R}$ such that $\varphi(u^\rho) = (\rho - d_2)/\mathbf{G}(1)$ (such a u^ρ exists, since, by assumption, $(\rho - d_2)/\mathbf{G}(1) \in \mathrm{im}\varphi$). Let $u \in F(\mathbb{Z}_+)$ be the unique solution of (3.1) and set $v := u - u^\rho\vartheta \in F(\mathbb{Z}_+)$. Moreover, we define $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}$ by,

$$\widetilde{\varphi}(\xi) := \varphi(\xi + u^\rho) - \varphi(u^\rho), \quad \forall\, \xi \in \mathbb{R}.$$

Then by (3.1),

$$v = u^0\vartheta - u^\rho\vartheta + kJ(\rho\vartheta - g - d) - kJ\varphi(u^\rho)G\vartheta - kJ(G(\widetilde{\varphi} \circ v)).$$

Hence it follows that

$$v = r - kJ(G(\widetilde{\varphi} \circ v)), \tag{3.2}$$

where the function r is given by

$$r = u^0\vartheta - u^\rho\vartheta - kJ(g + d_1 + \varphi(u^\rho)G\vartheta - (\rho - d_2)\vartheta). \tag{3.3}$$

In order to apply Theorem 2.1 to Eq. (3.2), we first show that $r \in m^2(\mathbb{Z}_+)$, that is, r satisfies the relevant assumption in Theorem 2.1. Since by assumption $Jg, Jd_1 \in m^2(\mathbb{Z}_+)$, we immediately see that

$$u^0\vartheta - u^\rho\vartheta - kJ(g + d_1) \in m^2(\mathbb{Z}_+).$$

Therefore, by (3.3), it is sufficient to show that the function

$$n \mapsto (J(\varphi(u^\rho)G\vartheta - (\rho - d_2)\vartheta))(n) = \varphi(u^\rho)(J(G\vartheta - \mathbf{G}(1)\vartheta))(n)$$

belongs to $m^2(\mathbb{Z}_+)$, where we have used that $\varphi(u^\rho) = (\rho - d_2)/\mathbf{G}(1)$. Note that

$$J(G\vartheta - \mathbf{G}(1)\vartheta) = J(G\vartheta - \mathbf{G}(1)\vartheta) - \mathbf{G}'(1)\vartheta + \mathbf{G}'(1)\vartheta. \tag{3.4}$$

By assumption (A′), $J(G\vartheta - \mathbf{G}(1)\vartheta) - \mathbf{G}'(1)\vartheta \in l^2(\mathbb{Z}_+)$. Hence it follows from (3.4) that $J(G\vartheta - \mathbf{G}(1)\vartheta) \in m^2(\mathbb{Z}_+)$.

Proof of Statement 1. Since φ is non-decreasing, locally Lipschitz continuous and satisfies $\varphi - \varphi(0) \in \mathscr{S}(a)$ for some $a \in (0, \infty)$, a routine argument shows that there exists $b \in (0, \infty)$ such that $\widetilde{\varphi} \in \mathscr{S}(b)$. Define $k^* := 1/|bf(G)| \in (0, \infty)$. Let $k \in (0, k^*)$ and set $\varepsilon := \frac{1}{2}(1/b + kf(G))$. Then $\varepsilon > 0$ and we can choose some $q > 0$ such that

$$\mathrm{essinf}_{\theta \in (0,2\pi)} \mathrm{Re}\left[\left(\frac{q}{e^{i\theta}} + \frac{1}{e^{i\theta} - 1}\right)\mathbf{G}(e^{i\theta})\right] \geq f(G) - \frac{\varepsilon}{k} = \frac{\varepsilon - 1/b}{k}.$$

Thus

$$\frac{1}{b} + \mathrm{Re}\left[\left(\frac{q}{e^{i\theta}} + \frac{1}{e^{i\theta} - 1}\right)k\mathbf{G}(e^{i\theta})\right] \geq \varepsilon, \quad \text{a.e. } \theta \in (0, 2\pi),$$

and (2.3) holds with a replaced by b and $\mathbf{G}$ replaced by $k\mathbf{G}$. An application of Theorem 2.1 to (3.2) now yields that $\lim_{n\to\infty} v(n)$ exists and is finite, $\widetilde{\varphi} \circ v \in l^2(\mathbb{Z}_+)$ and $\nabla v \in l^2(\mathbb{Z}_+)$. Consequently, we have $\lim_{n\to\infty} u(n) =: u^\infty$ exists and is finite,

$$\varphi(u^\infty) = \varphi(u^\rho) = (\rho - d_2)/\mathbf{G}(1) \quad \text{and} \quad \varphi \circ u - \varphi(u^\infty) \in l^2(\mathbb{Z}_+).$$

We note that $\Delta v = \Delta u$ and since $\nabla v \in l^2(\mathbb{Z}_+)$, we have $\Delta u \in l^2(\mathbb{Z}_+)$. Therefore, $e = \Delta u/k \in l^2(\mathbb{Z}_+)$, and hence, in particular, $\lim_{n\to\infty} e(n) = 0$.

Proof of Statement 2. Noting that, by hypothesis, φ is non-decreasing and globally Lipschitz with Lipschitz constant $\lambda > 0$, it is easy to show that $\widetilde{\varphi} \in \mathscr{S}(\lambda)$. Now the arguments in the proof of statement 1 apply with b replaced by λ. □

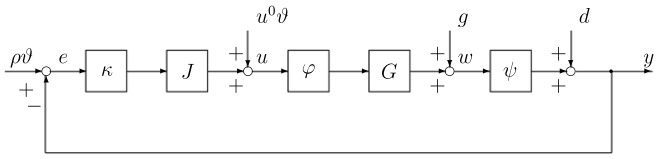

Fig. 4. Integral control in the presence of input and output nonlinearities.

3.2. Integral control in the presence of input and output nonlinearities

In this subsection, we generalize the feedback scheme considered in Section 3.1, to allow for a time-varying gain and nonlinearities in the output as well as in the input.

Consider the feedback system shown in Fig. 4, where $\kappa : \mathbb{Z}_+ \to \mathbb{R}$ is a time-varying gain, the operator $G \in \mathscr{B}(l^2(\mathbb{Z}_+))$ is shift-invariant with transfer function denoted by $\mathbf{G}$, $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ are static input and output nonlinearities, respectively, $\rho \in \mathbb{R}$ is a constant reference value, $u^0 \in \mathbb{R}$ is the initial state of the integrator (or, equivalently, the initial value of u), the function g models the effect of non-zero initial conditions of the system with input–output operator G and the function d is an external disturbance.

The feedback system shown in Fig. 4 is described by the equation

$$u = u^0\vartheta + J(\kappa(\rho\vartheta - d - \psi(g + G(\varphi \circ u)))). \tag{3.5}$$

Trivially, (3.5) is equivalent to the initial-value problem

$$(\Delta u)(n) = \kappa(n)(\rho - d(n) - \psi(g(n) + (G(\varphi \circ u))(n))), \quad n \in \mathbb{Z}_+, u(0) = u^0 \in \mathbb{R}. \tag{3.6}$$

It is easy to show that (3.5) has a unique solution $u \in F(\mathbb{Z}_+)$. The objective is to determine gain functions κ such that the tracking error

$$e(n) := \rho - y(n) = \rho - d(n) - \psi(g(n) + (G(\varphi \circ u))(n))$$

converges to 0 as $n \to \infty$.

We introduce the set of feasible reference values

$$\mathscr{R}(G, \varphi, \psi) := \{\psi(\mathbf{G}(1)v) : v \in \mathrm{cl}(\mathrm{im}\varphi)\}.$$

It is clear that $\mathscr{R}(G, \varphi, \psi)$ is an interval, provided that φ and ψ are continuous. It can be shown that if φ and ψ are continuous, then $\rho \in \mathscr{R}(G, \varphi, \psi)$ is close to being a necessary condition for tracking insofar as, if tracking of ρ is achievable, whilst maintaining boundedness of $\varphi \circ u$ and w, then $\rho \in \mathscr{R}(G, \varphi, \psi)$; see [15] for details.

We define,

$$f_0(G) := \mathrm{essinf}_{\theta\in(0,2\pi)} \mathrm{Re}\left[\frac{\mathbf{G}(e^{i\theta})}{e^{i\theta} - 1}\right].$$

If the transfer function $\mathbf{G}$ of G satisfies assumption (A), then $-\infty < f_0(G) \leq -\mathbf{G}(1)/2$, see [15].

Theorem 3.4. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+))$ be a shift-invariant operator with transfer function $\mathbf{G}$. Assume that assumption (A) holds with $\mathbf{G}(1) > 0$, $g \in l^2(\mathbb{Z}_+,)$, $d = d_1 + d_2\vartheta$ with $d_1 \in l^1(\mathbb{Z}_+)$, $d_2 \in \mathbb{R}$ and $n \mapsto \sum_{j=n}^{\infty} |d_1(j)| \in l^2(\mathbb{Z}_+)$, $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ are non-decreasing and globally Lipschitz continuous with Lipschitz constants $\lambda_1 > 0$ and $\lambda_2 > 0$, $\rho - d_2 \in \mathscr{R}(G, \varphi, \psi)$ and $\kappa : \mathbb{Z}_+ \to \mathbb{R}$ is bounded and non-negative with*

$$\limsup_{n\to\infty} \kappa(n) < 1/|\lambda_1\lambda_2 f_0(G)|. \tag{3.7}$$

Let $u : \mathbb{Z}_+ \to \mathbb{R}$ be the unique solution of (3.5). Then the following statements hold.

1. *The limit $(\varphi \circ u)^\infty := \lim_{n\to\infty} \varphi(u(n))$ exists and is finite and $\nabla(\varphi \circ u) \in l^2(\mathbb{Z}_+)$.*
2. *The signals $w = g + G(\varphi \circ u)$ and $y = \psi \circ w + d$ have finite limits satisfying*
$$\lim_{n\to\infty} w(n) = \mathbf{G}(1)(\varphi \circ u)^\infty, \quad \lim_{n\to\infty} y(n) = \psi(\mathbf{G}(1)(\varphi \circ u)^\infty) + d_2.$$
3. *If $\kappa \notin l^1(\mathbb{Z}_+)$, then $\lim_{n\to\infty} y(n) = \rho$, or equivalently, $\lim_{n\to\infty} e(n) = 0$.*
4. *If $\rho - d_2$ is an interior point of $\mathscr{R}(G, \varphi, \psi)$, then u is bounded.*

Remark 3.5. (i) A sufficient condition for d_1 to satisfy the assumptions in Theorem 3.4 is that the function $j \mapsto d_1(j)j^\alpha$ is in $l^2(\mathbb{Z}_+)$ for some $\alpha > 1$, see [15] for details.

(ii) Note that it is not necessary to know $f_0(G)$ or the constant λ in order to apply Theorem 3.4: if κ is chosen such that $\kappa(n) \to 0$ as $n \to \infty$ and $\kappa \notin l^1(\mathbb{Z}_+)$ (e.g. $\kappa(n) = (1 + n)^{-p}$ with $p \in (0, 1]$), then the conclusions of statement 3 hold. However, from a practical point of view, gain functions κ with $\lim_{n\to\infty} \kappa(n) = 0$ might not be appropriate, since the system essentially operates in open loop as $n \to \infty$.

(iii) In general d_2 is unknown, but it is reasonable to assume that $d_2 \in [\alpha, \beta]$, where α and β are known constants. The condition

$$\rho - \alpha, \rho - \beta \in \mathscr{R}(G, \varphi, \psi)$$

does not involve d_2 and is sufficient for $\rho - d_2$ to be in $\mathscr{R}(G, \varphi, \psi)$. Furthermore, if φ and ψ are continuous, $\lim_{\xi\to\pm\infty} \varphi(\xi) = \pm\infty$ and $\lim_{\xi\to\pm\infty} \psi(\xi) = \pm\infty$, then $\mathscr{R}(G, \varphi, \psi) = \mathbb{R}$ and it is clear that $\rho - d_2 \in \mathscr{R}(G, \varphi, \psi)$.

(iv) Theorem 3.4 can be viewed as a generalization of Theorem 3.2, allowing for input as well as output nonlinearities. However, since $|f_0(G)| \geq |f(G)|$, the range of integrator gains achieving asymptotic tracking is smaller in Theorem 3.4 than in Theorem 3.2.

Proof of Theorem 3.4. Let $u : \mathbb{Z}_+ \to \mathbb{R}$ be the unique solution of (3.5). We shall prove Theorem 3.4 by applying Corollary 2.4 to the equation satisfied by the input signal

$$w = g + G(\varphi \circ u)$$

of the output nonlinearity ψ, modified with an offset which depends on ρ and d_2. Since $\rho - d_2 \in \mathscr{R}(G, \varphi, \psi)$, there exists $\varphi^\rho \in \mathrm{cl}(\mathrm{im}\varphi)$ satisfying

$$\psi(\mathbf{G}(1)\varphi^\rho) = \rho - d_2.$$

Set

$$\widetilde{w} := w - \mathbf{G}(1)\varphi^\rho\vartheta = g + G(\varphi \circ u) - \mathbf{G}(1)\varphi^\rho\vartheta. \tag{3.8}$$

Furthermore, we define $\widetilde{\psi} : \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{\psi}(\xi) := \psi(\xi + \mathbf{G}(1)\varphi^\rho) - \rho + d_2, \quad \forall\, \xi \in \mathbb{R}.$$

Note that $\widetilde{\psi}(0) = 0$, and so, since ψ is non-decreasing and globally Lipschitz with Lipschitz constant λ_2, we have

$$0 \leq \widetilde{\psi}(\xi)\xi \leq \lambda_2\xi^2, \quad \forall\, \xi \in \mathbb{R}. \tag{3.9}$$

Using (3.6),

$$\Delta u = \kappa(\rho\vartheta - d - \psi(g + G(\varphi \circ u))) = \kappa(\rho\vartheta - d - \psi(\widetilde{w} + \mathbf{G}(1)\varphi^\rho)) = -\kappa(d_1 + \widetilde{\psi} \circ \widetilde{w}).$$

Defining

$$b_u(n) := \begin{cases} \dfrac{(\Delta(\varphi \circ u))(n)}{(\Delta u)(n)}, & \text{if } (\Delta u)(n) \neq 0, \\ 0, & \text{if } (\Delta u)(n) = 0, \end{cases}$$

it follows that,

$$\begin{aligned}(\Delta(\varphi \circ u))(n) = b_u(n)(\Delta u)(n) &= -b_u(n)\kappa(n)d_1(n) - b_u(n)\kappa(n)\widetilde{\psi}(\widetilde{w}(n)) \\ &= \widetilde{d}_1(n) - (N \circ \widetilde{w})(n), \quad \forall\, n \in \mathbb{Z}_+,\end{aligned} \tag{3.10}$$

where the function $N : \mathbb{Z}_+ \times \mathbb{R} \to \mathbb{R}$ is defined by

$$N(n, \xi) := b_u(n)\kappa(n)\widetilde{\psi}(\xi), \quad \forall\, (n, \xi) \in \mathbb{Z}_+ \times \mathbb{R}$$

and $\widetilde{d}_1 : \mathbb{Z}_+ \to \mathbb{R}$ is defined by

$$\widetilde{d}_1(n) := b_u(n)\kappa(n)d_1(n), \quad \forall\, n \in \mathbb{Z}_+.$$

Since φ is non-decreasing and globally Lipschitz with Lipschitz constant λ_1, we see that $0 \leq b_u(n) \leq \lambda_1$. Combining this with (3.9) yields

$$0 \leq N(n, \xi)\xi \leq \lambda_1\lambda_2\kappa(n)\xi^2, \quad \forall\, (n, \xi) \in \mathbb{Z}_+ \times \mathbb{R}. \tag{3.11}$$

It follows from (3.10) that,

$$\varphi \circ u = \varphi(u^0)\vartheta - J\widetilde{d}_1 - J(N \circ \widetilde{w}). \tag{3.12}$$

Applying G to both sides of (3.12), and using the fact that, by shift-invariance, G commutes with J, we obtain

$$G(\varphi \circ u) = \varphi(u^0)G\vartheta - GJ\widetilde{d}_1 - J(G(N \circ \widetilde{w})).$$

Invoking (3.8) yields

$$\widetilde{w} = r - J(G(N \circ \widetilde{w})), \tag{3.13}$$

where

$$r := g - \mathbf{G}(1)\varphi^\rho\vartheta + \varphi(u^0)G\vartheta - GJ\widetilde{d}_1.$$

Proof of Statement 1. Clearly, (3.13) is of the form (2.1) with u and φ replaced by $\widetilde{w}$ and N respectively. Therefore we may apply Corollary 2.4, provided the relevant assumptions are satisfied. By (3.7) there exists $a > 0$ satisfying

$$\lambda_1\lambda_2 \limsup_{n\to\infty} \kappa(n) < a < 1/|f_0(G)|. \tag{3.14}$$

By the definition of $f_0(G)$, there exists $\varepsilon > 0$ such that

$$\frac{1}{a} + \operatorname{Re} \frac{\mathbf{G}(e^{i\theta})}{e^{i\theta} - 1} \geq \varepsilon, \quad \text{a.e. } \theta \in (0, 2\pi).$$

Moreover, it follows from (3.11) and (3.14) that there exists $n_0 \geq 0$ such that

$$0 \leq N(n, \xi)\xi \leq a\xi^2, \quad \forall\, n \geq n_0, \ \forall\, \xi \in \mathbb{R}.$$

The above two inequalities show that (2.38) and (2.39) hold with φ, Q and P replaced by N, I and $1/a$, respectively. In order to apply Corollary 2.4, it remains to verify that $r \in m^2(\mathbb{Z}_+)$. To this end note that

$$r = g + \mathbf{G}(1)(\varphi(u^0)\vartheta - \varphi^\rho\vartheta) + \varphi(u^0)(G\vartheta - \mathbf{G}(1)\vartheta) - GJ\widetilde{d}_1. \tag{3.15}$$

Since b_u and κ are bounded, $d_1 \in l^1(\mathbb{Z}_+)$ and by assumption the function $n \mapsto \sum_{j=n}^{\infty} |d_1(j)| \in l^2(\mathbb{Z}_+)$, we deduce that $\widetilde{d}_1 = b_u\kappa d_1 \in l^1(\mathbb{Z}_+)$ and

$$n \mapsto \sum_{j=n}^{\infty} |\widetilde{d}_1(j)| \in l^2(\mathbb{Z}_+). \tag{3.16}$$

Since $\widetilde{d}_1 \in l^1(\mathbb{Z}_+)$, there exists $\sigma \in \mathbb{R}$ such that,

$$\lim_{n\to\infty} (J\widetilde{d}_1)(n) = \sigma.$$

From (3.16) we deduce that $J\widetilde{d}_1 - \sigma\vartheta \in l^2(\mathbb{Z}_+)$. We have,

$$GJ\widetilde{d}_1 = G(J\widetilde{d}_1 - \sigma\vartheta) + \sigma G\vartheta. \tag{3.17}$$

Noting that $G\vartheta = (G\vartheta - \mathbf{G}(1)\vartheta) + \mathbf{G}(1)\vartheta$ and that $G\vartheta - \mathbf{G}(1)\vartheta \in l^2(\mathbb{Z}_+)$ (since, by assumption (A), the function $z \mapsto (\mathbf{G}(z) - \mathbf{G}(1))/(z - 1)$ is in $H^2(\mathbb{E})$), we deduce that $G\vartheta \in m^2(\mathbb{Z}_+)$. Using this, the fact that $G \in \mathscr{B}(l^2(\mathbb{Z}_+))$ and $J\widetilde{d}_1 - \sigma\vartheta \in l^2(\mathbb{Z}_+)$, we obtain from (3.17) that $GJ\widetilde{d}_1 \in m^2(\mathbb{Z}_+)$. Since $G\vartheta - \mathbf{G}(1)\vartheta \in l^2(\mathbb{Z}_+)$ and, by assumption, $g \in l^2(\mathbb{Z}_+)$, we conclude from (3.15) that $r \in m^2(\mathbb{Z}_+)$. Invoking (3.12), an application of statement 4 of Theorem 2.2 to (3.13) now yields that

$$\Delta(\varphi \circ u) = -\widetilde{d}_1 - (N \circ \widetilde{w}) \in l^2(\mathbb{Z}_+),$$

and the limit

$$\lim_{n\to\infty} (\varphi \circ u)(n) = \varphi(u^0) - \sigma - \lim_{n\to\infty} (J(N \circ \widetilde{w}))(n)$$

exists and is finite, completing the proof of statement 1.

Proof of Statement 2. Using the shift-invariance of G, a simple calculation shows that

$$G(\varphi \circ u) = H(\Delta(\varphi \circ u)) + \varphi(u^0)(G\vartheta - \mathbf{G}(1)\vartheta) + \mathbf{G}(1)(\varphi \circ u), \tag{3.18}$$

where $H := JG - \mathbf{G}(1)J$. Since $H \in \mathscr{B}(l^2(\mathbb{Z}_+))$ and $\Delta(\varphi \circ u) \in l^2(\mathbb{Z}_+)$, it follows that $H(\Delta(\varphi \circ u)) \in l^2(\mathbb{Z}_+)$. Furthermore, $G\vartheta - \mathbf{G}(1)\vartheta \in l^2(\mathbb{Z}_+)$ and $(\varphi \circ u)^\infty = \lim_{n\to\infty}(\varphi \circ u)(n)$ exists and is finite (by statement 1). Consequently, we deduce from (3.18) that

$$\lim_{n\to\infty} (G(\varphi \circ u))(n) = \mathbf{G}(1)(\varphi \circ u)^\infty.$$

Since $w = g + G(\varphi \circ u)$ and $g \in l^2(\mathbb{Z}_+)$, we obtain

$$\lim_{n\to\infty} w(n) = \mathbf{G}(1)(\varphi \circ u)^\infty.$$

Consequently,

$$y^\infty := \lim_{n\to\infty} y(n) = \lim_{n\to\infty} (\psi \circ w)(n) + d_2 = \psi(\mathbf{G}(1)(\varphi \circ u)^\infty) + d_2,$$

where we have used that $d_1(n) \to 0$ as $n \to \infty$.

Proof of Statement 3. We know by statement 2 that $y(n) \to y^\infty$ as $n \to \infty$. Seeking a contradiction, suppose that $y^\infty < \rho$ (the case $y^\infty > \rho$ can be treated in an analogous way). Setting $\varepsilon := \rho - y^\infty > 0$ and by taking $n_0 \geq 0$ large enough, we

have $\rho - y(n) \geq \varepsilon/2$ whenever $n \geq n_0$. Hence we see that,

$$(\Delta u)(n) = \kappa(n)(\rho - y(n)) \geq \kappa(n)\frac{\varepsilon}{2}, \quad \forall\, n \geq n_0.$$

Summing the above inequality from n_0 to $m-1$ gives

$$u(m) \geq u(n_0) + \frac{\varepsilon}{2}\sum_{k=n_0}^{m-1}\kappa(k) \to \infty, \quad m \to \infty.$$

Consequently, since φ is non-decreasing,

$$\overline{\varphi} := \sup_{v\in\mathbb{R}}\varphi(v) = (\varphi\circ u)^{\infty}.$$

Hence, by statement 2, $y^{\infty} = \psi(\mathbf{G}(1)\overline{\varphi}) + d_2$. Since $\rho - d_2 \in \mathscr{R}(G,\varphi,\psi)$ (by assumption) and using the fact that ψ is non-decreasing and $\mathbf{G}(1) > 0$, we obtain

$$\rho \leq \sup\mathscr{R}(G,\varphi,\psi) + d_2 = \psi(\mathbf{G}(1)\overline{\varphi}) + d_2 = y^{\infty},$$

contradicting the supposition that $y^{\infty} < \rho$. Setting $\underline{\varphi} := \inf_{v\in\mathbb{R}}\varphi(v)$, an analogous argument shows that if $y^{\infty} > \rho$, then necessarily $y^{\infty} = \psi(\mathbf{G}(1)\underline{\varphi}) + d_2$, which likewise leads to a contradiction since $\rho \geq \psi(\mathbf{G}(1)\underline{\varphi}) + d_2$.

Proof of Statement 4. By statement 1, the limit $(\varphi\circ u)^{\infty} = \lim_{n\to\infty}(\varphi\circ u)(n)$ exists and is finite. We consider the following two cases.

Case 1: $\kappa \notin l^1(\mathbb{Z}_+)$.

If $\kappa \notin l^1(\mathbb{Z}_+)$, then by statements 2 and 3, $\rho - d_2 = \psi(\mathbf{G}(1)(\varphi\circ u)^{\infty})$. Unboundedness of u would imply that there exists a sequence (v_n) with $\lim_{n\to\infty}|v_n| = \infty$ and such that,

$$\rho - d_2 = \lim_{n\to\infty}\psi(\mathbf{G}(1)\varphi(v_n)).$$

Since the function $v \mapsto \psi(\mathbf{G}(1)\varphi(v))$ is non-decreasing, this would in turn yield $\rho - d_2 = \sup\mathscr{R}(G,\varphi,\psi)$ or $\rho - d_2 = \inf\mathscr{R}(G,\varphi,\psi)$, showing that u must be bounded if $\rho - d_2$ is an interior point of $\mathscr{R}(G,\varphi,\psi)$.

Case 2: $\kappa \in l^1(\mathbb{Z}_+)$.

By statement 2 we know that $\rho\vartheta - y$ is bounded. With $\kappa \in l^1(\mathbb{Z}_+)$ it now follows that $\kappa(\rho\vartheta - y) \in l^1(\mathbb{Z}_+)$. Since $\Delta u = \kappa(\rho\vartheta - y)$, we conclude that u is bounded. □

If in the feedback equations (3.1) and (3.5), the integrator J is replaced by $J_I = J + I$ (cf. Remark 2.5), then results similar to Theorems 3.2 and 3.4 can be proved (see [15] for details).

Finally, Theorems 3.2 and 3.4 constitute the discrete-time counterpart of the continuous-time low-gain integral control theory developed in [6,12]. Related discrete-time integral control results can be found in [16,17]: however, whilst similar in spirit, the relevant results in [16,17] are less general than Theorems 3.2 and 3.4 (for example, the choice of integrator gain is more restricted, the underlying linear system is assumed to be regular, output nonlinearities are not included). Moreover, the treatment in [16,17] is based exclusively on state-space methods (assuming power stability of the linear subsystem) in contrast to the input–output methodology adopted in Sections 2 and 3 of the current paper.

4. Applications to discrete-time state-space systems

This section is devoted to applications of the results in Sections 2 and 3 to infinite-dimensional discrete-time state-space systems. Consider the discrete-time system

$$x(n+1) = Ax(n) + Bv(n), \quad x(0) = x^0 \in X, \tag{4.1a}$$

$$w(n) = Cx(n) + Dv(n), \tag{4.1b}$$

with state-space X (a Banach space), input space U (a Hilbert space), output space U, $A \in \mathscr{B}(X)$, $B \in \mathscr{B}(U,X)$, $C \in \mathscr{B}(X,U)$ and $D \in \mathscr{B}(U)$.

The solution x of (4.1a) is given by

$$x(n) = \begin{cases} A^n x^0 + \displaystyle\sum_{j=0}^{n-1} A^{n-1-j}Bv(j), & n \geq 1, \\ x^0, & n = 0. \end{cases} \tag{4.2}$$

The formula for the input–output operator G of (4.1) is

$$(Gv)(n) = \begin{cases} \displaystyle\sum_{j=0}^{n-1} CA^{n-1-j}Bv(j) + Dv(n), & n \geq 1, \\ Dv(0), & n = 0, \end{cases} \quad \forall\, v \in F(\mathbb{Z}_+, U). \tag{4.3}$$

Let $\mathbf{G}$ denote the transfer function of the system (4.1). Then, for $|z|$ greater than the spectral radius of A,

$$\mathbf{G}(z) = C(zI - A)^{-1}B + D.$$

The operator A is said to be *power stable* if there exist $M > 0$ and $\mu \in (0, 1)$ such that $\|A^n\| \leq M\mu^n$ for all $n \in \mathbb{Z}_+$. By definition, the system (4.1) is *power stable* if A is power stable. We say that system (4.1) is *strongly stable* if the following four conditions are satisfied:

(i) G is l^2-stable, that is, $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$, or, equivalently, the transfer function $\mathbf{G} \in H^\infty(\mathbb{E}, \mathscr{B}(U))$.
(ii) A is strongly stable, i.e., for all $\xi \in X$, $A^n\xi \to 0$ as $n \to \infty$.
(iii) There exists $\alpha \geq 0$ such that,

$$\left\| \sum_{j=0}^{\infty} A^j B v(j) \right\| \leq \alpha \|v\|_{l^2}, \quad \forall\, v \in l^2(\mathbb{Z}_+, U). \tag{4.4}$$

(iv) There exists $\beta \geq 0$ such that,

$$\left(\sum_{j=0}^{\infty} \|CA^j\xi\|^2 \right)^{1/2} \leq \beta \|\xi\|, \quad \forall\, \xi \in X. \tag{4.5}$$

Trivially, if system (4.1) is power stable, then system (4.1) is strongly stable. The converse is not true: a counterexample can be found in [15].

Remark 4.1. If system (4.1) is strongly stable and $1 \in \varrho(A)$, then $\mathbf{G} \in H^\infty(\mathbb{E}, \mathscr{B}(U))$ and $\mathbf{G}$ extends analytically to a neighbourhood of 1, so that assumption (A′) (and hence (A)) is satisfied.

To obtain state-space versions of the results in Sections 2 and 3, it is useful to state the following two lemmas, the proofs of which can be found in the Appendix.

Lemma 4.2. *Assume that A is strongly stable and (4.4) holds. Then there exists $K \geq 0$ such that, for all $x^0 \in X$ and $v \in l^2(\mathbb{Z}_+, U)$, the solution x of (4.1a) satisfies*

$$\|x\|_{l^\infty} \leq K(\|x^0\| + \|v\|_{l^2}).$$

Moreover, $\lim_{n\to\infty} x(n) = 0$.

Lemma 4.3. *Assume that A is strongly stable, $1 \in \varrho(A)$ and (4.4) holds. Let $v \in F(\mathbb{Z}_+, U)$ be such that $\nabla v \in l^2(\mathbb{Z}_+, U)$. Then for all $x^0 \in X$, the solution x of (4.1a) satisfies*

$$\lim_{n\to\infty} (x(n) - (I - A)^{-1}Bv(n)) = 0.$$

In the following, for the sake of brevity, we give state-space interpretations of Theorems 2.2 and 3.4 only. Theorems 2.1 and 3.2 have similar state-space interpretations.

Let $\varphi : \mathbb{Z}_+ \times U \to U$ be a (time-dependent) static nonlinearity and consider system (4.1), with input nonlinearity $v = \varphi \circ u$ in feedback interconnection with the integrator $\Delta u = -w$, that is,

$$x(n + 1) = Ax(n) + B(\varphi \circ u)(n), \quad x(0) = x^0 \in X, \tag{4.6a}$$

$$u(n + 1) = u(n) - Cx(n) - D(\varphi \circ u)(n), \quad u(0) = u^0 \in U, \tag{4.6b}$$

where as before, by slight abuse of notation, $\varphi \circ u$ denotes the function $n \mapsto \varphi(n, u(n))$.

The following result is a state-space version of Theorem 2.2.

Theorem 4.4. *Assume that system (4.1) is strongly stable, $1 \in \varrho(A)$ and $\mathbf{G}(1)$ is invertible. Let $\varphi : \mathbb{Z}_+ \times U \to U$ be a nonlinearity. Suppose there exist self-adjoint $P \in \mathscr{B}(U)$, invertible $Q \in \mathscr{B}(U)$ with $Q\mathbf{G}(1) = [Q\mathbf{G}(1)]^* \geq 0$ and a number $\varepsilon > 0$ such that (2.31) and (2.32) hold. Let (x, u) be the unique solution of (4.6). Then there exists a constant $K \geq 0$ (which depends only on (A, B, C, D), Q, P and ε, but not on u^0 and x^0) such that*

$$\|x\|_{l^\infty} + \|u\|_{l^\infty} + \|\nabla u\|_{l^2} + \|\varphi \circ u\|_{l^2} + (\|\mathrm{Re}\,\langle(\varphi \circ u), Qu\rangle\|_{l^1})^{1/2} + \sup_{n\geq 0} \left\| \sum_{j=0}^{n} (\varphi \circ u)(j) \right\|$$

$$\leq K(\|x^0\| + \|u^0\|). \tag{4.7}$$

Moreover, $\lim_{n\to\infty} x(n) = 0$ and statements 2–6 of Theorem 2.2 hold with $r_2 = u^0 - C(I - A)^{-1}x^0$ in (2.34) and (2.35).

Note that (4.7) implies in particular that the equilibrium $(0, 0)$ of (4.6) is stable in the large. Furthermore, under the additional assumption that conditions (B)–(D) of Theorem 2.2 hold, it follows from Theorem 4.4 that the equilibrium $(0, 0)$ of (4.6) is asymptotically stable, provided that there exists a neighbourhood $V \subset U$ of 0 such that $\varphi^{-1}(0) \cap V = \{0\}$ and globally asymptotically stable, provided that $\varphi^{-1}(0) = \{0\}$.

Proof of Theorem 4.4. Let (x, u) be the unique solution of (4.6). It follows from (4.6b) that

$$u = u^0\vartheta - J(Cx + D(\varphi \circ u)).$$

Invoking (4.2) and (4.3), we see that u satisfies

$$u = r - J(G(\varphi \circ u)), \tag{4.8}$$

where $r \in F(\mathbb{Z}_+, U)$ is defined by

$$r(n) := \begin{cases} u^0 - \sum_{j=0}^{n-1} CA^j x^0, & n \geq 1, \\ u^0, & n = 0. \end{cases}$$

In order to apply Theorem 2.2 to (4.8), we need to verify the relevant assumptions. It is clear (see Remark 4.1), that $\mathbf{G}$ satisfies assumption (A). To show that $r \in m^2(\mathbb{Z}_+, U)$, we first note that,

$$\sum_{j=0}^{n-1} CA^j x^0 = C(I - A)^{-1}x^0 - CA^n(I - A)^{-1}x^0, \quad \forall\, n \geq 1.$$

Hence

$$r(n) = u^0 - C(I - A)^{-1}x^0 + CA^n(I - A)^{-1}x^0, \quad \forall\, n \in \mathbb{Z}_+.$$

Writing $r = r_1 + r_2\vartheta$, where $r_1(n) := CA^n(I-A)^{-1}x^0$ for all $n \in \mathbb{Z}_+$ and $r_2 := u^0 - C(I-A)^{-1}x^0$, we see that $r \in m^2(\mathbb{Z}_+, U)$, since, by (4.5), $r_1 \in l^2(\mathbb{Z}_+, U)$. Thus, an application of Theorem 2.2 shows that there exists a constant $K_1 \geq 0$ (not depending on r) such that

$$\|u\|_{l^\infty} + \|\nabla u\|_{l^2} + \|\varphi \circ u\|_{l^2} + (\|\mathrm{Re}\,\langle(\varphi \circ u), Qu\rangle\|_{l^1})^{1/2} + \sup_{n\geq 0}\left\|\sum_{j=0}^{n}(\varphi \circ u)(j)\right\| \leq K_1\|r\|_{m^2}. \tag{4.9}$$

It also follows immediately that statements 2–6 of Theorem 2.2 hold with $r_2 = u^0 - C(I - A)^{-1}x^0$ in (2.34) and (2.35). Furthermore, since $\varphi \circ u \in l^2(\mathbb{Z}_+, U)$, it follows from Lemma 4.2 that $x \in l^\infty(\mathbb{Z}_+, U)$, $\lim_{n\to\infty} x(n) = 0$ and

$$\|x\|_{l^\infty} \leq K_2(\|x^0\| + \|\varphi \circ u\|_{l^2}), \tag{4.10}$$

for some suitable constant $K_2 \geq 0$ (not depending on x^0 and u). Moreover, using (4.5) we deduce

$$\begin{aligned} \|r\|_{m^2} &= \left(\sum_{j=0}^{\infty}\|CA^j(I - A)^{-1}x^0\|^2\right)^{1/2} + \|u^0 - C(I - A)^{-1}x^0\| \\ &\leq \beta\|(I - A)^{-1}x^0\| + \|u^0 - C(I - A)^{-1}x^0\|, \end{aligned}$$

for some $\beta \geq 0$. Therefore,

$$\|r\|_{m^2} \leq K_3(\|x^0\| + \|u^0\|), \tag{4.11}$$

for some suitable constant $K_3 \geq 0$ (not depending on x^0 and u^0). Combining (4.9)–(4.11) shows that there exists a constant $K \geq 0$ (not depending on x^0 and u^0) such that (4.7) holds. □

Next we present a state-space version of Theorem 3.4. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an input and $\psi : \mathbb{R} \to \mathbb{R}$ be an output nonlinearity, let $\rho \in \mathbb{R}$ be a reference value, $\kappa : \mathbb{Z}_+ \to \mathbb{R}$ a time-varying gain, d an external disturbance, and consider the discrete-time system (4.1), with input nonlinearity $v = \varphi \circ u$, in feedback interconnection with the integrator $\Delta u = \kappa(\rho\vartheta - d - \psi \circ w)$, that is,

$$x(n + 1) = Ax(n) + B(\varphi \circ u)(n), \quad x(0) = x^0 \in X, \tag{4.12a}$$

$$w(n) = Cx(n) + D(\varphi \circ u)(n), \tag{4.12b}$$

$$u(n + 1) = u(n) + \kappa(n)(\rho - d(n) - (\psi \circ w)(n)), \quad u(0) = u^0 \in \mathbb{R}. \tag{4.12c}$$

We define the tracking error $e := \rho - d - \psi \circ w$.

Theorem 4.5. *Assume that system (4.1) is strongly stable, $1 \in \varrho(A)$ and $\mathbf{G}(1) > 0$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ be non-decreasing and globally Lipschitz continuous with Lipschitz constants $\lambda_1 > 0$ and $\lambda_2 > 0$. Suppose that $d = d_1 + d_2\vartheta$ with $d_1 \in l^1(\mathbb{Z}_+)$, $d_2 \in \mathbb{R}$ and $n \mapsto \sum_{j=n}^{\infty}|d_1(j)| \in l^2(\mathbb{Z}_+)$, $\rho - d_2 \in \mathscr{R}(G, \varphi, \psi)$ and $\kappa : \mathbb{Z}_+ \to \mathbb{R}$ is bounded and non-negative with*

$$\limsup_{n\to\infty}\kappa(n) < 1/|\lambda_1\lambda_2 f_0(G)|.$$

Let (x, u) be the unique solution of (4.12). Then the following statements hold.

1. *The limit $(\varphi \circ u)^\infty := \lim_{n\to\infty} \varphi(u(n))$ exists and is finite, $\nabla(\varphi \circ u) \in l^2(\mathbb{Z}_+)$ and $\lim_{n\to\infty} x(n) = (I - A)^{-1}B(\varphi \circ u)^\infty$.*
2. *The signals w and $y := \psi \circ w + d$ have finite limits satisfying*
$$\lim_{n\to\infty} w(n) = \mathbf{G}(1)(\varphi \circ u)^\infty, \quad \lim_{n\to\infty} y(n) = \psi(\mathbf{G}(1)(\varphi \circ u)^\infty) + d_2.$$
3. *If $\kappa \notin l^1(\mathbb{Z}_+)$, then $\lim_{n\to\infty} y(n) = \rho$, or equivalently,*
$$\lim_{n\to\infty} e(n) = 0.$$
4. *If $\rho - d_2$ is an interior point of $\mathscr{R}(G, \varphi, \psi)$, then u is bounded.*

Proof. Let (x, u) be the unique solution of (4.12). As a consequence of (4.12b) and (4.12c) we have that

$$u = u^0\vartheta + J(\kappa(\rho\vartheta - d - \psi \circ (Cx + D(\varphi \circ u)))).$$

Invoking (4.2) and (4.3), we see that u satisfies

$$u = u^0\vartheta + J(\kappa(\rho\vartheta - d - \psi(g + G(\varphi \circ u)))), \tag{4.13}$$

where $g(n) := CA^n x^0$. In order to apply Theorem 3.4 to (4.13), we need to verify the relevant assumptions. It is clear (see Remark 4.1) that $\mathbf{G}$ satisfies assumption (A). By (4.5)

$$\left(\sum_{j=0}^{\infty} \|CA^j x^0\|^2\right)^{1/2} \leq \beta\|x^0\|,$$

for some $\beta \geq 0$, hence $g \in l^2(\mathbb{Z}_+)$. Statements 2–4 now follow immediately from the application of Theorem 3.4 to (4.13). It remains to show that statement 1 holds. Again applying Theorem 3.4 to (4.13), we obtain that $(\varphi \circ u)^\infty := \lim_{n\to\infty} \varphi(u(n))$ exists and is finite and $\nabla(\varphi \circ u) \in l^2(\mathbb{Z}_+)$. Lemma 4.3 shows that $\lim_{n\to\infty} x(n) = (I - A)^{-1}B(\varphi \circ u)^\infty$. □

Appendix

Proof of Lemma 4.2. By (4.2) we have,

$$\|x(n)\| \leq \|A^n\|\|x^0\| + \left\|\sum_{j=0}^{n-1} A^{n-1-j}Bv(j)\right\|, \quad \forall\, n \geq 1. \tag{A.1}$$

Defining

$$\widetilde{v}(j) := \begin{cases} v(n-1-j), & \text{if } 0 \leq j \leq n-1, \\ 0, & \text{if } j \geq n, \end{cases}$$

it follows from (4.4) that,

$$\left\|\sum_{j=0}^{n-1} A^{n-1-j}Bv(j)\right\| = \left\|\sum_{j=0}^{\infty} A^j B\widetilde{v}(j)\right\| \leq \alpha\|\widetilde{v}\|_{l^2} \leq \alpha\|v\|_{l^2}, \quad \forall\, n \geq 1, \tag{A.2}$$

for some $\alpha \geq 0$. Noting that A is strongly stable, an application of the uniform boundedness principle to $\{A^n\}_{n\in\mathbb{N}}$ yields that $\|A^n\| \leq K_1$ for some constant $K_1 \geq 0$ and for all $n \in \mathbb{Z}_+$. Therefore, by (A.1) and (A.2),

$$\|x(n)\| \leq K_1\|x^0\| + \alpha\|v\|_{l^2}, \quad \forall\, n \in \mathbb{Z}_+.$$

Consequently, with $K := \max\{K_1, \alpha\}$,

$$\|x\|_{l^\infty} \leq K(\|x^0\| + \|v\|_{l^2}).$$

It remains to show that $\lim_{n\to\infty} x(n) = 0$. To this end note that, by (4.1a),

$$x(n) = A^{n-m}x(m) + \sum_{j=m}^{n-1} A^{n-1-j}Bv(j), \quad \forall\, n \geq m+1. \tag{A.3}$$

Defining

$$\widetilde{v}(j) := \begin{cases} v(n-1-j), & 0 \leq j \leq n-m-1, \\ 0, & j \geq n-m, \end{cases}$$

it follows that

$$\sum_{j=m}^{n-1} A^{n-1-j}Bv(j) = \sum_{j=0}^{n-m-1} A^j B\widetilde{v}(j), \quad \forall\, n \geq m+1.$$

Invoking (4.4), we obtain

$$\left\| \sum_{j=m}^{n-1} A^{n-1-j} B v(j) \right\| = \left\| \sum_{j=0}^{n-m-1} A^j B \widetilde{v}(j) \right\| \leq \alpha \left(\sum_{j=m}^{n-1} \|v(j)\|^2 \right)^{1/2}, \quad \forall\, n \geq m+1, \tag{A.4}$$

for some $\alpha \geq 0$. Let $\varepsilon > 0$. Since $v \in l^2(\mathbb{Z}_+, U)$, there exists $m \geq 0$ such that

$$\sum_{j=m}^{n-1} \|v(j)\|^2 \leq \sum_{j=m}^{\infty} \|v(j)\|^2 \leq \frac{\varepsilon^2}{4\alpha^2}, \quad \forall\, n \geq m+1. \tag{A.5}$$

By strong stability of A, there exists $m_1 \geq 1$ such that

$$\|A^n x(m)\| \leq \varepsilon/2, \quad \forall\, n \geq m_1. \tag{A.6}$$

Combining (A.3)–(A.6), we obtain

$$\|x(n)\| \leq \|A^{n-m} x(m)\| + \left\| \sum_{j=m}^{n-1} A^{n-1-j} B v(j) \right\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall\, n \geq m + m_1,$$

showing that $\lim_{n\to\infty} \|x(n)\| = 0$. □

Proof of Lemma 4.3. By hypothesis, $1 \in \varrho(A)$, and thus, by (4.1a),

$$(I-A)^{-1} x(n+1) = (I-A)^{-1} A x(n) + (I-A)^{-1} B v(n). \tag{A.7}$$

Noting that $(I-A)^{-1}A = (I-A)^{-1} - I$, we obtain from (A.7),

$$(I-A)^{-1} (\Delta x)(n) = -(x(n) - (I-A)^{-1} B v(n)). \tag{A.8}$$

Moreover, by (4.1a),

$$(\Delta x)(n+1) = A(\Delta x)(n) + B(\Delta v)(n). \tag{A.9}$$

Since, by assumption, $\Delta v \in l^2(\mathbb{Z}_+, U)$, an application of Lemma 4.2 to (A.9) yields

$$\lim_{n\to\infty} (\Delta x)(n) = 0.$$

Consequently, taking the limit as $n \to \infty$ in (A.8), we conclude that

$$\lim_{n\to\infty} (x(n) - (I-A)^{-1} B v(n)) = 0. \quad \square$$

References

[1] C.A. Desoer, M. Vidyasagar, Feedback Systems: Input–Output Properties, Academic Press, New York, 1975.
[2] H.K. Khalil, Nonlinear Systems, 3rd edn, Prentice-Hall, Upper Saddle River, NJ, 2002.
[3] M. Vidyasagar, Nonlinear Systems Analysis, 2nd edn, Prentice Hall, Englewood Cliffs, NJ, 1993.
[4] V.A. Yakubovich, G.A. Leonov, A.Kh. Gelig, Stability of Stationary Sets in Control Systems with Discontinuous Nonlinearities, World Scientific, New Jersey, 2004.
[5] A. Halanay, V. Răsvan, Stability and Stable Oscillations in Discrete-Time Systems, Gordon and Breach Science Publishers, Amsterdam, 2000.
[6] R.F. Curtain, H. Logemann, O. Staffans, Stability results of Popov-type for infinite-dimensional systems with applications to integral control, Proc. London Math Soc. 86 (2003) 779–816.
[7] W.M. Haddad, D.S. Bernstein, Explicit construction of quadratic Lyapunov functions for the small gain, positivity, circle and Popov theorems and their application to robust stability. Part II: Discrete-time theory, Internat. J. Robust Nonlinear Control 4 (1994) 249–265.
[8] R.P. Iwens, A.R. Bergen, Frequency-criteria for bounded-input–bounded-output stability of nonlinear sampled-data systems, IEEE Trans. Automat. Control 12 (1967) 46–53.
[9] M. Larsen, P.V. Kokotović, A brief look at the Tsypkin criterion: From analysis to design, Internat. J. Adapt. Control Signal Process. 15 (2001) 121–128.
[10] G.P. Szegö, J.B. Pearson, On the absolute stability of sampled-data systems: The 'indirect control' case, IEEE Trans. Automat. Control 9 (1964) 160–163.
[11] H. Logemann, S. Townley, Discrete-time low-gain control of uncertain infinite-dimensional systems, IEEE Trans. Automat. Control 42 (1997) 22–37.
[12] T. Fliegner, H. Logemann, E.P. Ryan, Low-gain integral control of well-posed linear infinite-dimensional systems with input and output nonlinearities, J. Math. Anal. Appl. 261 (2001) 307–336.
[13] J.J. Coughlan, H. Logemann, Sampled-data integral control of infinite-dimensional systems with actuator and sensor nonlinearities: An input–output approach (in preparation).
[14] J.R. Partington, Linear Operators and Linear Systems, Cambridge University Press, Cambridge, 2004.
[15] J.J. Coughlan, Absolute stability results for infinite-dimensional discrete-time systems with applications to sampled-data integral control, Ph.D. Thesis, University of Bath, 2007 (available at http://www.maths.bath.ac.uk/~hl/THESES/coughlan_thesis.pdf).
[16] H. Logemann, A.D. Mawby, Discrete-time and sampled-data low-gain integral control of infinite-dimensional linear systems in the presence of input hysteresis, SIAM J. Control Optim. 41 (2002) 113–140.
[17] H. Logemann, E.P. Ryan, Time-varying and adaptive discrete-time low-gain control of infinite-dimensional linear systems subject to input nonlinearities, Math. Control Signals Systems 13 (2000) 293–317.