BOUNDARY INTEGRAL EQUATION METHODS FOR HIGH FREQUENCY SCATTERING PROBLEMS

Technical Introduction to the Project

There is huge mathematical and engineering interest in acoustic and electromagnetic wave scattering problems, driven by many applications such as modelling radar, sonar, acoustic noise barriers, atmospheric particle scattering, ultrasound and VLSI. For problems in infinite domains and media which are predominantly homogeneous, the boundary element method is a very popular solver, used in a number of large commercial codes, see e.g. [16]. In many practical applications the characteristic length scale L of the domain is large compared to the wavelength λ. Then the small dimensionless wavelength λ/L induces oscillatory solutions, and conventional (piecewise polynomial) boundary elements for this multiscale problem yield full matrices of dimension at least $N = (L/\lambda)^{d-1}$ (in $\mathbb{R}^d$). (Domain finite elements lead to sparse matrices but require even larger N.) Since this "loss of robustness" as $L/\lambda \to \infty$ puts high frequency problems outside the reach of many standard algorithms, much recent research has been devoted to finding more robust methods. **The fundamental problem addressed in this research programme is how to solve problems with indefinitely high L/λ using finite computing resources.**

One approach is to seek faster implementations of standard methods. Fast multipole methods have allowed conventional BEM solutions for much larger N (e.g. [17,18]), but it remains impossible to compute with L/λ much beyond a few hundred in 3D. To allow larger L/λ, a highly promising new direction is the development of "hybrid" algorithms, which incorporate asymptotic information about the oscillation of the solution into the approximation space [1,4,5,7,11,13-15,19-22,25]. Initial experiments using geometric-optics type approximations on simple model problems indicate the possibility of delivering almost uniform accuracy for N fixed as $L/\lambda \to \infty$. This has generated considerable international interest (e.g. a meeting at Banff (Canada), and a 2007 programme at the Newton Institute). However rigorous theory and applications to real-world (non-smooth, 3D) problems, especially where non-geometrical optics (diffractive) effects are significant, will require substantial new advances. **This proposal presents an ambitious programme at the interface between numerical analysis, applied analysis and scientific computing, developing both the mathematics and the implementation of these methods so that they become viable practical tools.** The strategic need for more research at this interface was highlighted in reports of the **UK International Reviews of Mathematics** (2004, p. 25) and **High-Performance Computing** (2005, p. 11). The practical relevance of our programme is underlined by the involvement of our project partners (BAE Systems, Institute of Cancer Research, The Met Office and Schlumberger), whose financial commitment will ensure their staff can take part in this project.

Suppose an incident plane wave $u^I(x) = \exp(\mathrm{i}kx.\hat{a})$, with unit direction $\hat{a}$ and k denoting wavenumber ($k = 2\pi/\lambda$), is scattered by a bounded object Ω to produce a radiating scattered wave u^S. Then the total wave $u = u^I + u^S$ satisfies the Helmholtz equation:

$$\Delta u + k^2 u = 0 \quad \text{in} \quad D := \mathbb{R}^d \setminus \Omega \quad (d = 2 \text{ or } 3). \tag{1}$$

Using the fundamental solution (in 3D $\Phi(x,y) := \exp(\mathrm{i}k|x-y|)/4\pi|x-y|$), in the simplest Dirichlet case ($u = 0$ on the boundary Γ) the scattering problem can be reformulated as the boundary integral equation

$$\frac{\partial u}{\partial n}(x) + \int_\Gamma \left(\frac{\partial \Phi(x,y)}{\partial n(x)} + \mathrm{i}\eta\Phi(x,y) \right) \frac{\partial u}{\partial n}(y)\, \mathrm{d}s(y) = f(x), \quad x \in \Gamma, \tag{2}$$

where $\partial/\partial n$ denotes normal derivative, $\eta > 0$ is a *coupling parameter* (which ensures that (2) is well-posed) f is determined by the incident wave and $\partial u/\partial n$ is to be found. Standard boundary element methods approximate the whole (oscillatory) $\partial u/\partial n$ by (piecewise) polynomials. By contrast the *hybrid methods* which we shall propose use approximations like

$$\frac{\partial u}{\partial n}(x) \approx \sum_{m=1}^{M} k \exp(\mathrm{i}k\gamma_m(x)) V_m(x,k), \quad x \in \Gamma, \tag{3}$$

where the phase functions $\gamma_m(x)$ are chosen *a priori* and only the unknowns $V_m(x,k)$ are approximated (with respect to x) by piecewise polynomials. The key point is that asymptotic analysis can be used to determine the γ_m in such a way that the V_m are very much less oscillatory than the original $\partial u/\partial n$.

For a smooth convex obstacle it was argued non-rigorously in [13] that if one takes $M = 1$ and $\gamma_1(x) = x \cdot \hat{a}$ (the geometric optics ansatz) in (3), then approximating $V_1(\cdot, k)$ with a finite element space of dimension $N = O((kL)^{(d-1)/3})$ should preserve accuracy as $k \to \infty$. Recognising that this complexity growth arises from the breakdown of the geometric optics ansatz, [14] employed mesh grading in $O(k^{-1/3})$ neighbourhoods of the shadow boundaries and obtained experimental k-robust convergence on smooth model problems. In this project a mathematical understanding of these methods, and the generation of better methods, will be obtained exploiting "uniform asymptotics", which allow a more complete description of complex oscillatory behaviour than is provided by simple geometric optics.

Much of the rigorous analysis on hybrid methods to date has been provided by the groups at Reading and Bath, generating significant international interest. In [7] the authors considered the same smooth convex case as in [13,14] in 2D and proved k-explicit results on the continuity and coercivity of the boundary integral operator in (2) and then used (3) with $M = 1$ and different polynomial approximations for V_1 near the shadow boundaries and in the illuminated zone. They proved that the numerical error essentially remains bounded as $k \to \infty$, provided the number of degrees of freedom grows like $k^{1/9}$. This estimate required extensions of the very delicate uniform asymptotics in e.g. [24]. Separate work [1,4] on scattering by a convex polygon shows that in this case the appropriate specific form of (3) is

$$\frac{\partial u}{\partial n}(x) \approx P.O. + k \sum_{m=1}^{M} [\exp(\mathrm{i}kx \cdot \hat{a}_m) V_m^+(x,k) + \exp(-\mathrm{i}kx \cdot \hat{a}_m) V_m^-(x,k)], \quad x \in \Gamma, \tag{4}$$

where $P.O.$ is the physical optics approximation, M is the number of sides of the polygon, the unit vector $\hat{a}_m$ is parallel to the mth side, and the function $V_m^{\pm}$ is assumed non-zero only on side m. (Physically the terms in the summation represent the diffracted field.) It is shown in [4] that, if $V_m^{\pm}$ are approximated by piecewise polynomials on carefully chosen graded meshes, then uniformly accurate approximations are obtained as $k \to \infty$, provided the number of degrees of freedom grows like $O(\log^{3/2} k)$. The results [1,4] are inspired by earlier results at Reading [5,11] which proposed an algorithm for scattering by a half plane with piecewise continuous impedance boundary condition and proved its k-independent accuracy and complexity, the first such result for any scattering algorithm.

Recent developments for the non-convex case include [19], where scattering by configurations of several smooth convex scatterers is approximated by a Neumann series. Each term in the series is computed by solving a single scattering problem, using the ansatz (3), with the phase factors γ_m determined by ray-tracing. While the convergence of the Neumann series is analysed in [19], and the numerical results are impressive, the proof of $k-$ robustness of the overall algorithm remains a challenging open problem.

The results mentioned above just scratch the surface of this emerging area. This project will combine cutting-edge techniques in asymptotic and numerical analysis to design new wavenumber robust algorithms, will realise these algorithms in efficient software and will apply them to industrially-relevant problems.

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