A new multiscale finite element method for high-contrast elliptic interface problems

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Oberwolfach, August 2008

Main points

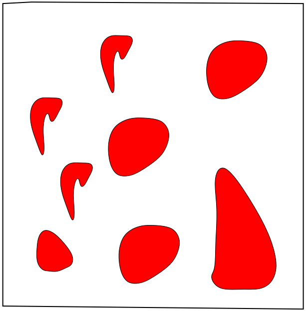
- Elliptic interface problems (**jumping coefficients**)
- **MSFE:** Solve local homogeneous PDEs for basis functions
- **New result:** methods with **optimal convergence** independent of the **contrast** even with “naive meshing”.
- Method involves **new interior boundary conditions** for basis functions.
- Theory involves **new regularity results** for elliptic interface problems
- Method is a **generalisation** of the P1-continuous Galerkin method.
- Theory at present in 2D only.

elliptic interface problems

Find $u \in H_0^1(\Omega)$:

$$\int_\Omega \mathcal{A}(x)\nabla u(x).\nabla v(x)dx = \int_\Omega F(x)v(x)dx , \quad v \in H_0^1(\Omega) ,$$

"High contrast" piecewise constant coefficient $\mathcal{A}$:



Inclusions: $\Omega_1, \ldots, \Omega_m$ $\quad \Omega_0 = \Omega \backslash \cup_{i=1}^m \Omega_i$. $\quad$ Interface Γ.

Scale by $\mathcal{A}_{\min} = \min_x \mathcal{A}(x)$: Find $u \in H_0^1(\Omega)$ such that

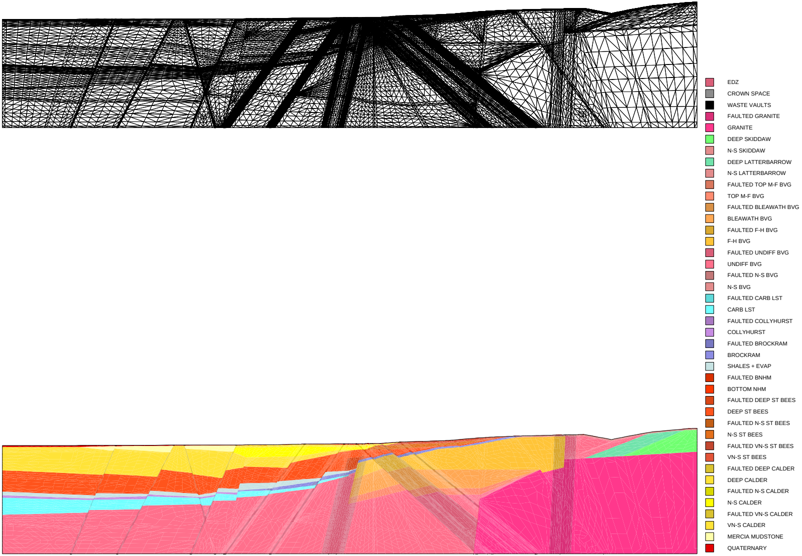
$$a(u,v) \ := \int_\Omega \alpha(x)\nabla u(x).\nabla v(x)dx \ = \ (f,v)_{L_2(\Omega)} \ , \quad v \in H_0^1(\Omega) \ ,$$

with

$$\alpha(x) = \frac{1}{\mathcal{A}_{\min}}\mathcal{A}(x) \ , \quad f(x) = \frac{1}{\mathcal{A}_{\min}}F(x) \ .$$

Then $\alpha(x) \ \geq \ 1$ and the difficulty is characterised by the **contrast**, a **large parameter**

$$\hat{\alpha} := \frac{\max_x \mathcal{A}(x)}{\min_x \mathcal{A}(x)} \geq 1 \ .$$


EDZ
CROWN SPACE
WASTE VAULTS
FAULTED GRANITE
GRANITE
DEEP SKIDDAW
N-S SKIDDAW
DEEP LATTERBARROW
N-S LATTERBARROW
FAULTED TOP M-F BVG
TOP M-F BVG
FAULTED BLEAWATH BVG
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FAULTED F-H BVG
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FAULTED UNDIFF BVG
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FAULTED N-S BVG
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CARB LST
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BROCKRAM
SHALES + EVAP
FAULTED BNHM
BOTTOM NHM
FAULTED DEEP ST BEES
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QUATERNARY

Asymptotic cases

Case I: $\widehat{\alpha} := \min_{i=1,\ldots,m} \alpha_i \to \infty\ , \quad \alpha_0 = 1$

Highly permeable inclusions in hardly permeable matrix

Case II: $\widehat{\alpha} := \alpha_0 \to \infty\ , \quad \max_{i=1,\ldots,m} \alpha_i \ \leq \ \mathrm{Const.}$

Hardly permeable inclusions in highly permeable matrix.

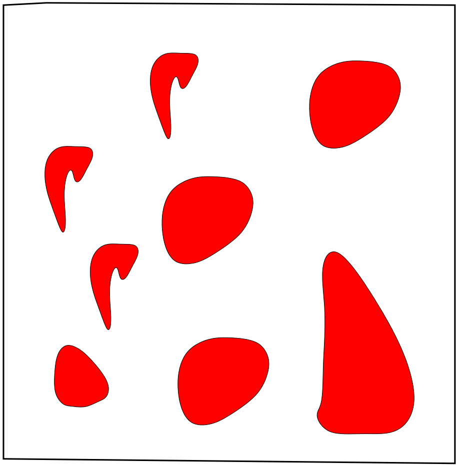
Regularity of solution:

Across an interface Γ separating Ω_- and Ω_+:

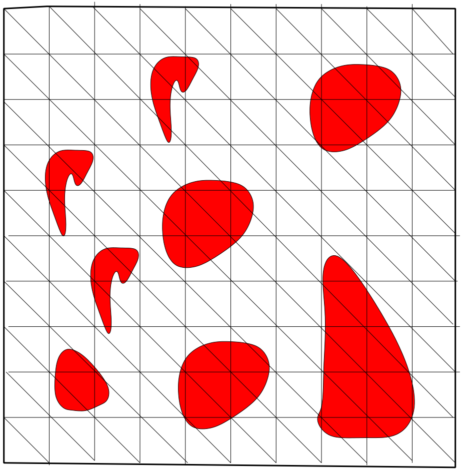
$$\alpha_- \frac{\partial u_-}{\partial n} = \alpha_+ \frac{\partial u_+}{\partial n}$$

Hence $u \in H^{3/2-\epsilon}(\Omega)$. For smooth problems $u \in H^2(\Omega)$

Naive meshing



Naive meshing



Use linear finite elements on the naive mesh.

Finite element space $\mathcal{V}$.

Naive error estimate Standard linear FEM: seek $u_h \in \mathcal{V}$:

$$a(u_h, v_h) = (f, v_h)_{L^2(\Omega)} \ , \quad v_h \in \mathcal{V}_h \ .$$

Energy norm: $|v|_{H^1(\Omega),\alpha} = \left\{\int_\Omega \alpha |\nabla v|^2\right\}^{1/2}$

$$\begin{aligned} |u - u_h|_{H^1(\Omega),\alpha} &\leq |u - \mathcal{I}_h u|_{H^1(\Omega),\alpha} \\ &\leq Ch^{1/2-\epsilon}|u|_{H^{3/2-\epsilon}(\Omega)} = C(\hat{\alpha})h^{1/2-\epsilon} \end{aligned}$$

($\mathcal{I}_h$ = piecewise linear interpolant):

Duality argument:

$$\|u - u_h\|_{L_2(\Omega)} \leq C'(\hat{\alpha})h^{1-2\epsilon} \ .$$

cf. robust $\mathcal{O}(h)$ and $\mathcal{O}(h^2)$ for smooth problems.

Analysis of $C(\hat{\alpha})$ and $C'(\hat{\alpha})$ requires delicate **regularity theory** (explicit in the **contrast** $\hat{\alpha}$).

"Multiscale" Finite Element Methods

Special finite element space: $\mathcal{V}^{\mathrm{MS}} = \mathrm{span}\{\Phi_p^{\mathrm{MS}}\}$, with

$\Phi_p^{\mathrm{MS}}(x_q) = \delta_{p,q}$ and, for all elements τ:

$$\begin{array}{lll} \Phi_p^{\mathrm{MS}}|\tau & \text{is linear} & \tau \cap \Gamma = \emptyset, \\ \Phi_p^{\mathrm{MS}}|\tau & \text{solves (*)} & \tau \cap \Gamma \neq \emptyset, \end{array}$$

Local Homogeneous Problems for the basis functions:

$$\int_\tau \alpha \nabla \Phi_p^{\mathrm{MS}} . \nabla v \;=\; 0 \;, \quad \text{for all } v \in H_0^1(\tau) \qquad (*)$$

Boundary conditions and subgrid **subgrid approximation** needed for (*).

MSFEM: seek $u_h^{\mathrm{MS}} \in \mathcal{V}^{\mathrm{MS}}$:

$$a(u_h^{\mathrm{MS}}, v_h^{\mathrm{MS}}) = (f, v_h^{\mathrm{MS}})_{L^2(\Omega)} \;, \quad v_h^{\mathrm{MS}} \in \mathcal{V}^{\mathrm{MS}} \;.$$

The main result

Theorem Assume

- Ω is a convex polygon
- the interface Γ is sufficiently smooth.
- $f \in H^{1/2+\epsilon}(\Omega)$.
- mesh sequence is quasiuniform

Then there exists a choice of boundary condition for Φ_p^{MS} such that

$$
\begin{aligned}
(i) \quad |u - u_h^{\mathrm{MS}}|_{H^1(\Omega),\alpha} &\lesssim h \left[h|f|^2_{H^{1/2+\epsilon}(\Omega)} + \|f\|^2_{L_2(\Omega)} \right]^{1/2}, \\
(ii) \quad \|u - u_h^{\mathrm{MS}}\|_{L_2(\Omega)} &\lesssim h^{2-\epsilon} \left[h|f|^2_{H^{1/2}(\Omega)} + \|f\|^2_{L_2(\Omega)} \right]^{1/2}.
\end{aligned}
$$

Hidden constants are independent of h and $\hat{\alpha}$.
requires slightly more regularity on f
non-standard duality argument for the L_2 estimate.

Multiscale methods

"Subgrid modelling", e.g. LES in turbulence models, modelling convective storms in NWF, etc..

Hughes et. al. 1995... **Hou and Wu, JCP 1997**:

$$-\nabla.a(x/\epsilon)\nabla u = f \quad \text{with } a \text{ periodic, smooth}$$

$$\|u - u_h^{\mathrm{MS}}\|_{H^1(\Omega)} \le C_1 h \|f\|_{L_2(\Omega)} + C_2(\epsilon/h)^{1/2} \quad \epsilon << h \ .$$

Many related papers, **Abdulle and E, 03, E & Engquist 04, Efendiev, Hou and Wu, 00, Arbogast & Boyd 06...** **Proofs of accuracy by homgenization arguments** but computations on general media

Preconditioning using multiscale coarsening:
IGG, Lechner & Scheichl 07, IGG & Scheichl 07, Pechstein and Scheichl 08. **Proofs of condition number estimates without homogenization structure**.

High contrast diffusion

Solvers and adaptivity (**Without homogenization structure:**)

- Many papers on domain decomposition methods (e.g. **Chan and Mathew, 04**) solvers independent of coefficient jumps (coarse mesh resolves discontinuity).
- Recent proof or robustness of geometric multigrid, **Xu and Zhu, 07**.
- Spectral clustering even without coarse resolution **IGG and Hagger 99, Vuik et. al 00, Aksoylu, IGG, Klie, Scheichl, 08**
- Robustness of a posteriori error estimators, e.g. **Bernardi and Verfürth 00, Ainsworth 05, Vohralik 08.**

Analysis of accuracy of underlying methods ??

Analysis of MSFE: The Main Idea

Optimality:

$$|u - u_h^{\rm MS}|_{H^1(\Omega),\alpha} \;\le\; |E_h^{\rm MS}|_{H^1(\Omega),\alpha} \,,$$

MS interpolant $\mathcal{I}_h^{\rm MS}$

$$E_h^{\rm MS} := u - \mathcal{I}_h^{\rm MS} \,.$$

By **definition** of multiscale basis functions:

$$a_\tau(E_h^{\rm MS}, v) \;=\; a_\tau(u, v) \;=\; (f, v)_{L^2(\tau)} \,, \quad \text{for all} \quad v \in H_0^1(\tau) \,.$$

Simple energy argument:

$$|E_h^{\rm MS}|_{H^1(\tau),\alpha} \;\lesssim\; |\widetilde{E}_h^{\rm MS}|_{H^1(\tau),\alpha} \;+\; h_\tau \|f\|_{L_2(\tau)} \,,$$

where $\widetilde{E}_h^{\rm MS} = E_h^{\rm MS}$ on $\partial\tau$ (**any extension**). Then

$$|E_h^{\rm MS}|^2_{H^1(\Omega),\alpha} \;\lesssim\; h^2 \left[\sum_\tau h_\tau^{-2} |\widetilde{E}_h^{\rm MS}|^2_{H^1(\tau),\alpha} + \|f\|^2_{L_2(\Omega)} \right] \,.$$

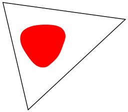
Seek BC s.t. there exists $\widetilde{E}_h^{\rm MS}$ yielding

$$\sum_\tau h_\tau^{-2} |\widetilde{E}_h^{\rm MS}|^2_{H^1(\tau),\alpha} \;\lesssim\; h |f|^2_{H^{1/2+\epsilon}(\Omega)} \,.$$

A simple application: Inclusion inside element

Example: $\hat{\alpha}$ **in interior**

1 in exterior (Ω_0)



Linear BC's and define $\widetilde{E}_h^{\mathrm{MS}} = \begin{cases} E_h^{\mathrm{MS}} & \text{on } \partial\tau \\ 0 & \text{on } \Gamma \end{cases}$.

Trace theorem (scaled):

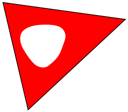
$$
\begin{aligned}
|\widetilde{E}_h^{\mathrm{MS}}|_{H^1(\tau),\alpha} &\lesssim h_\tau^{-1/2}\|E_h^{\mathrm{MS}}\|_{L_2(\partial\tau)} + h_\tau^{1/2}|E_h^{\mathrm{MS}}|_{H^1(\partial\tau)} \\
&\lesssim h_\tau^{3/2}\,\|D_t^2 u\|_{L^2(\partial\tau)} \quad \textbf{tangential derivative} \\
&\lesssim h_\tau^{3/2} \sum_{e\in\mathcal{E}(\tau)} \|D_e^2 u\|_{H^{1/2+\epsilon}(\tau\cap\Omega_0)}
\end{aligned}
$$

So
$$
\begin{aligned}
\sum_\tau h_\tau^{-2}|\widetilde{E}_h^{\mathrm{MS}}|^2_{H^1(\tau),\alpha} &\lesssim \sum_\tau h_\tau \sum_{e\in\mathcal{E}(\tau)} \|D_e^2 u\|^2_{H^{1/2+\epsilon}(\tau\cap\Omega_0)} \\
(\text{as } \hat{\alpha}\to\infty) \quad &\lesssim h\|f\|^2_{H^{1/2+\epsilon}(\Omega)}
\end{aligned}
$$

Regularity Theory!!

A simple application: Inclusion inside element

Example: $\hat{\alpha}$ **in exterior (Ω_0)**
1 in interior



Linear BC's and define $\widetilde{E}_h^{\mathrm{MS}} = \begin{cases} E_h^{\mathrm{MS}} & \text{on} \quad \partial\tau \\ 0 & \text{on} \quad \Gamma \end{cases}$.

Trace theorem (scaled):

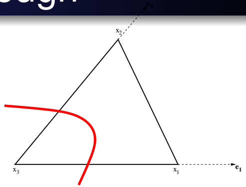
$$
\begin{aligned}
|\widetilde{E}_h^{\mathrm{MS}}|_{H^1(\tau),\alpha} &\lesssim h_\tau^{-1/2}\hat{\alpha}\|E_h^{\mathrm{MS}}\|_{L_2(\partial\tau)} + h_\tau^{1/2}\hat{\alpha}|E_h^{\mathrm{MS}}|_{H^1(\partial\tau)} \\
&\lesssim h_\tau^{3/2}\ \hat{\alpha}\ \|D_t^2 u\|_{L^2(\partial\tau)} \quad \textbf{tangential derivative} \\
&\lesssim h_\tau^{3/2} \sum_{e\in\mathcal{E}(\tau)} \hat{\alpha}\ \|D_e^2 u\|_{H^{1/2+\epsilon}(\tau\cap\Omega_0)}
\end{aligned}
$$

So
$$
\begin{aligned}
\sum_\tau h_\tau^{-2}|\widetilde{E}_h^{\mathrm{MS}}|^2_{H^1(\tau),\alpha} &\lesssim \sum_\tau h_\tau \sum_{e\in\mathcal{E}(\tau)} \hat{\alpha}\ \|D_e^2 u\|^2_{H^{1/2+\epsilon}(\tau\cap\Omega_0)} \\
(\text{as} \quad \hat{\alpha}\to\infty) \quad &\lesssim h\|f\|^2_{H^{1/2+\epsilon}(\Omega)} \quad \textbf{Regularity Theory!!}
\end{aligned}
$$

Solution flat in regions of high conductivity. (Proof?)

Much more complicated: "cutting through"

Generic case:



Looking for **piecewise linear** boundary condition: Simple Taylor expansion on true solution u:

$$r_i^+ \frac{\partial u^+}{\partial e_i}(y_i) + r_i^- \frac{\partial u^-}{\partial e_i}(y_i) = u(x_3) - u(x_1) + \mathcal{O}(h^2) \quad i = 1, 2$$

For gradients $\mathbf{g}_i$ of basis functions on two intersected edges, solve :

$$\begin{aligned} \mathbf{r}_i^T \mathbf{g}_i &= f_i \quad , \; i = 1, 2 \; , \\ \mathbf{g}_1 &= B(\theta_1, \theta_2, \beta, \hat{\alpha}) \mathbf{g}_2 \end{aligned}$$

Matrix B depends on angles of intersection, corner and $\hat{\alpha}$

If Γ orthogonal to edges, B reduces to **two independent interface conditions**. ("oscillatory b.c." of Hou and Wu (97)).

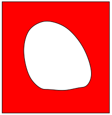
interface cutting through

- The recipe leads to **non-conforming** elements, but **averaging** returns conformity without loss of convergence.
- In conforming case $\mathrm{supp}(\Phi_p^{\mathrm{MS}})$ can grow with one extra layer of triangles
- Convergence theorem as before:

$$
\begin{aligned}
(i) \quad & |u - u_h^{\mathrm{MS}}|_{H^1(\Omega),\alpha} && \lesssim h \left[h |f|^2_{H^{1/2+\epsilon}(\Omega)} + \|f\|^2_{L_2(\Omega)} \right]^{1/2}, \\
(ii) \quad & \|u - u_h^{\mathrm{MS}}\|_{L_2(\Omega)} && \lesssim h^{2-\epsilon} \left[h |f|^2_{H^{1/2}(\Omega)} + \|f\|^2_{L_2(\Omega)} \right]^{1/2}.
\end{aligned}
$$

Regularity theory

Smooth well-separated inclusions
Particular case: picture



$$\begin{aligned} -\nabla.\alpha\nabla u &= f \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Theorem (**I.V. Kamotski and V.P. Smyshlyaev**)

$$|u|_{H^{2+s}(\Omega_0)} \lesssim \frac{1}{\hat{\alpha}}\|f\|_{H^s(\Omega)} \quad s \geq 0 \qquad (1)$$

$$|u|_{H^{2+s}(\Omega_1)} \lesssim \|f\|_{H^s(\Omega)} \quad s \geq 0 \qquad (2)$$

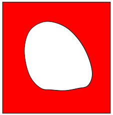
cf. Huang and Zou, J. Differential Equations, 2002

Idea of proof: Introduce $\widehat{u}$ solution of

$$-\nabla.\alpha_i\nabla\widehat{u} = f_i \ , \quad \text{on} \quad \Omega_i, \ i = 0, 1 \ , \quad \hat{u} = 0 \quad \text{on} \quad \partial\Omega, \Gamma$$

decoupled problems, $\widehat{u}$ satisfies estimates!

Consider remainder: $\widetilde{u} := u - \widehat{u}$:



$$-\Delta \widetilde{u}_i \;=\; 0 \quad \text{on } \Omega_1 \text{ and } \Omega_0 \quad \text{and} \quad \widetilde{u} = 0 \quad \text{on} \quad \partial\Omega$$

Jump condition on interface $\Gamma = \partial\Omega_1$:

$$\hat{\alpha}\frac{\partial \widetilde{u}_0}{\partial n} - \frac{\partial \widetilde{u}_1}{\partial n} \;=\; F \quad := \quad \frac{\partial \widehat{u}_1}{\partial n} - \hat{\alpha}\frac{\partial \widehat{u}_0}{\partial n} \qquad (\dagger)$$

Let $\widetilde{v} := \widetilde{u}|_\Gamma$ and introduce **Dirichlet to Neumann maps** $\mathcal{N}_i$

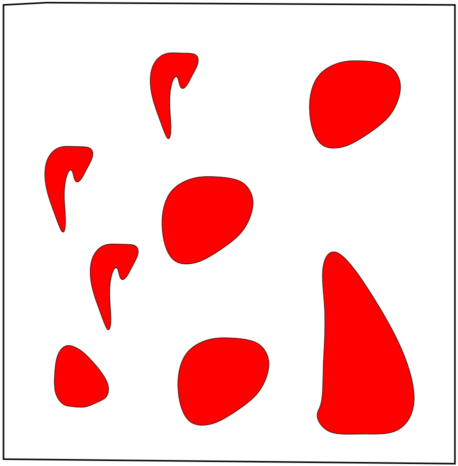
$$\begin{aligned} (\dagger) \iff \quad (\hat{\alpha}\mathcal{N}_0 - \mathcal{N}_1)\widetilde{v} \;&=\; F \\ \iff \quad (I - \hat{\alpha}^{-1}\mathcal{N}_0^{-1}\mathcal{N}_1)\widetilde{v} \;&=\; \hat{\alpha}^{-1}\mathcal{N}_0^{-1}F \end{aligned}$$

Contraction mapping ($\hat{\alpha}^{-1} \to 0$):

$$\begin{aligned} \|\widetilde{v}\|_{H^{s+3/2}(\Gamma)} \;&\lesssim\; \hat{\alpha}^{-1}\|\mathcal{N}_0^{-1}F\|_{H^{s+3/2}(\Gamma)} \lesssim \hat{\alpha}^{-1}\|F\|_{H^{s+1/2}(\Gamma)} \\ &\lesssim\; \hat{\alpha}^{-1}\|\widehat{u}_1\|_{H^{s+2}(\Omega_1)} + \|\widehat{u}_0\|_{H^{s+2}(\Omega_1)} \lesssim \hat{\alpha}^{-1}\|f\|_{H^s(\Omega)} \end{aligned}$$

(In this case $\|\tilde{u}\|_{H^{2+s}(\Omega_0)} \;=\; \mathcal{O}(\hat{\alpha}^{-1})$)

Slightly harder case:



Dirichlet to Neumann maps not invertible on “floating” domains.

Numerical Results

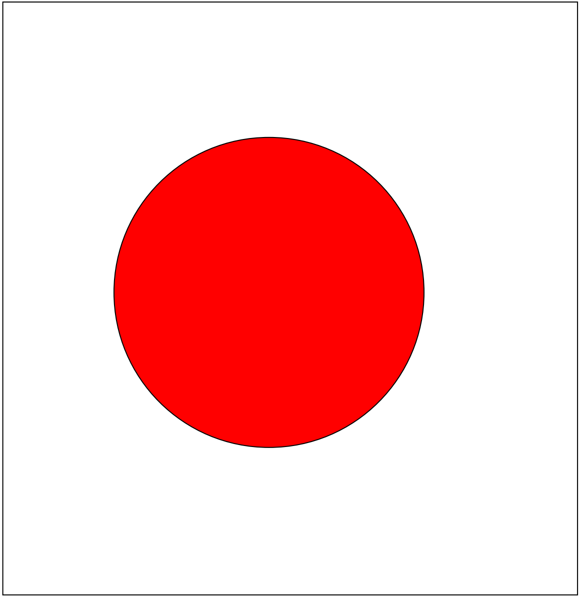
$$\begin{aligned} -\nabla.\alpha\nabla u &= f \quad \text{on} \quad \Omega := [0,1]^2, \\ u &= g \quad \text{on} \quad \partial\Omega \end{aligned}$$

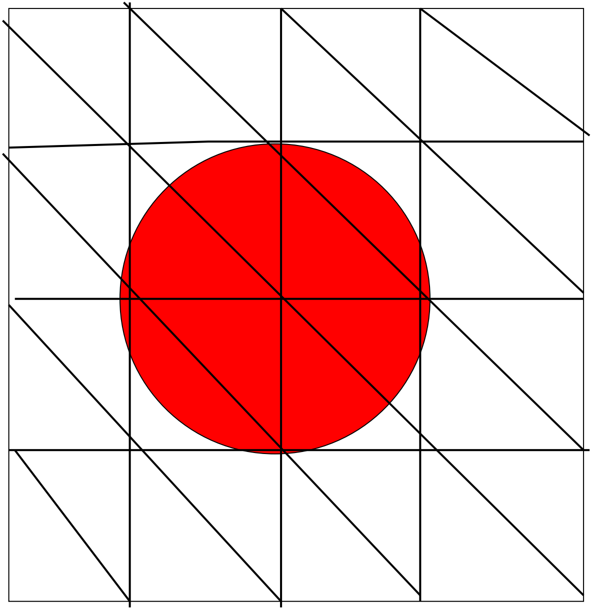
Interface is a circle of radius r_0,

$$\alpha(x) = \begin{cases} \alpha_1, & r < r_0 \\ \alpha_0, & r \geq r_0 \end{cases}$$

Exact solution:

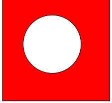
$$u(x) = u(r,\theta) = \begin{cases} \frac{r^3}{\alpha_1} & r < r_0 \\ \frac{r^3}{\alpha_0} + \left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0}\right) r_0^3 & r \geq r_0 \end{cases}$$





$\alpha_1 = 1, \quad \alpha_0 = \hat{\alpha} \to \infty$

(Impermeable inclusion in high permeable matrix)



H^1 seminorm errors:

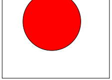
h	$\hat{\alpha} = 10$	$\hat{\alpha} = 10^3$	$\hat{\alpha} = 10^5$
1/8	2.55e-1	2.51e-1	2.54e-1
1/16	1.33e-1	1.24e-1	1.24e-1
1/32	6.22e-2	6.15-2	6.14e-2
1/64	3.26e-2	3.15e-2	3.07e-2
rate	1.0	1.0	1.0

L_2 errors:

h	$\hat{\alpha} = 10$	$\hat{\alpha} = 10^3$	$\hat{\alpha} = 10^5$
1/8	2.27e-2	2.27e-2	2.29e-2
1/16	5.75e-3	5.76e-3	5.78e-3
1/32	1.45e-3	1.45e-3	1.45e-3
1/64	3.73e-4	3.67e-4	3.63e-4
rate	1.98	1.98	1.99

$\alpha_0 = 1, \quad \alpha_1 = \hat{\alpha} \to \infty$

(Highly permeable inclusion in impermeable matrix)



H^1 seminorm errors:

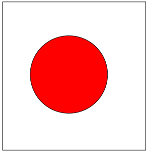
h	$\hat{\alpha} = 10$	$\hat{\alpha} = 10^3$	$\hat{\alpha} = 10^5$
1/8	1.09e-1	5.81e-2	5.90e-2
1/16	4.57e-2	2.75e-2	2.77e-2
1/32	1.43e-2	1.30e-2	1.27e-2
1/64	1.01e-2	6.52e-3	6.10e-3
rate	1.11	1.00	1.09

L_2 errors:

h	$\hat{\alpha} = 10$	$\hat{\alpha} = 10^3$	$\hat{\alpha} = 10^5$
1/8	4.83e-3	3.89-3	3.89e-3
1/16	1.32e-3	1.10e-3	1.10e-3
1/32	3.32e-4	2.91e-4	2.91e-4
1/64	8.73e-5	7.56e-5	7.53e-5
rate	1.92	1.88	1.88

Solution of subgrid problems

$h = 1/128$, $\alpha_1 = \hat{\alpha}$, $\alpha_0 = 1$



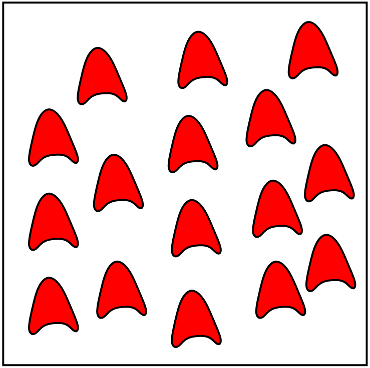
Subgrid problems solved by **Immersed finite element method** (Li, Lin, Wu (2003)).

L^2 errors, where M = # of subgrid elements

M	$\hat{\alpha}$ = 10	$\hat{\alpha} = 10^3$	$\hat{\alpha} = 10^5$
16	1.0307e-4	3.8052e-4	1.4611e-4
64	9.7183e-5	3.8131e-4	7.0831e-5
256	9.6699e-5	3.8188e-4	6.5619e-5
1024	9.6818e-5	3.8216e-4	6.4345e-5

Extensions under construction

Distance between inclusions and distance of inclusions from the boundary are “bad parameters”



With I. Kamotski and V.P. Smyshlyaev (Bath): inclusions separated by $\mathcal{O}(\epsilon)$ and diameter $\mathcal{O}(\epsilon)$.

Conclusion: Summary of results

- Elliptic interface problems with complicated interfaces have **irregular solutions** depending on contrast and interface
- Application of standard FE technoology will require **complicated mesh adaptivity** to resolve difficulties
- MSFE can **resolve** these difficulties on **“naive” meshes**.
- The **extra cost** is the solution of subgrid problems on some elements
- Analysis helps explain success of MSFE outside the homogenization framework.
- Regularity theory may help with analysis of **standard methods**.