Neutron Transport Theory: The diffusion approximation

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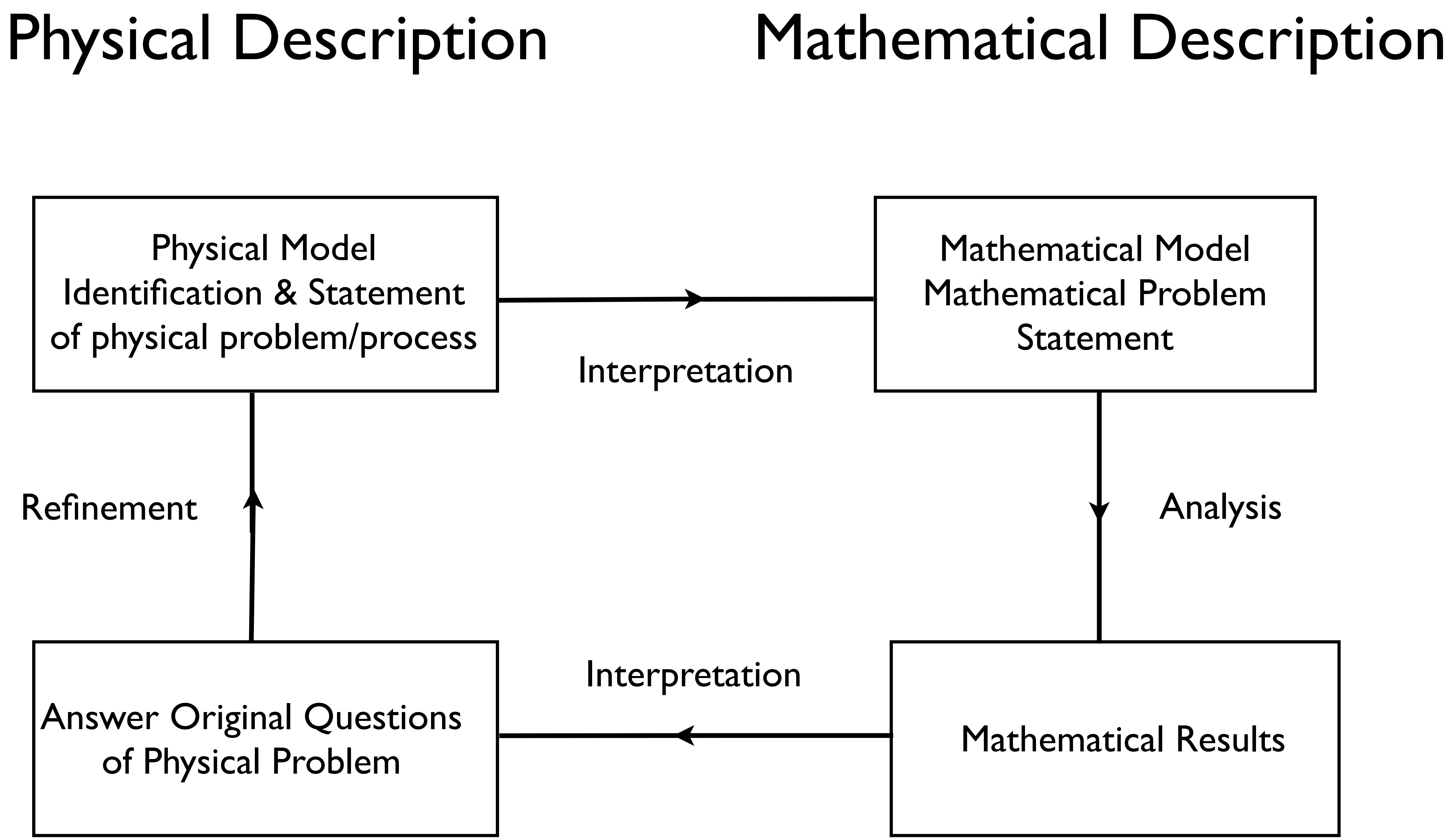
Outline

1. Modelling Process

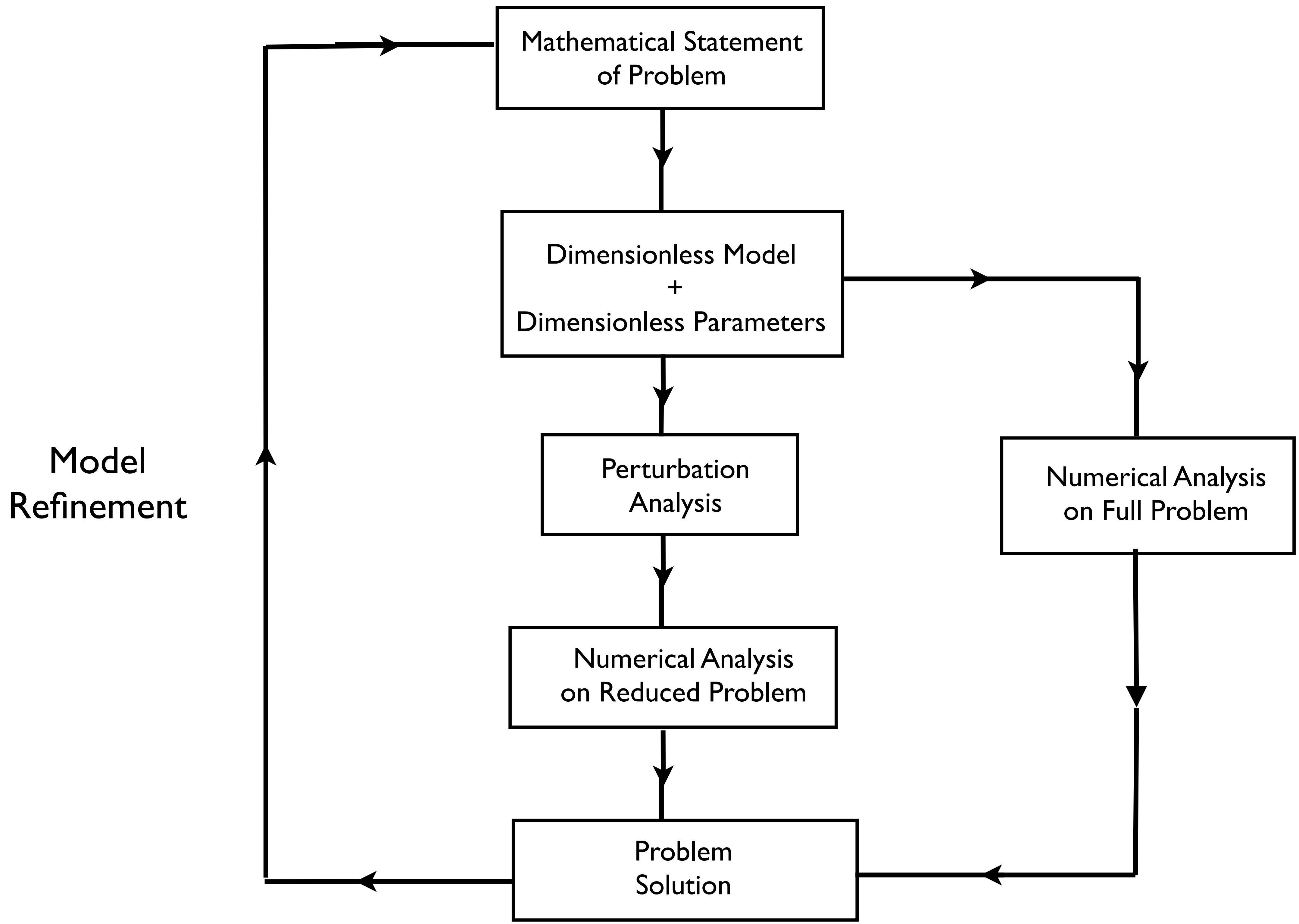
2. Neutron Transport

    1. Problem Formulation
    2. Non-dimensionalisation
    3. Asymptotic Analysis - (limit of small mean free path)

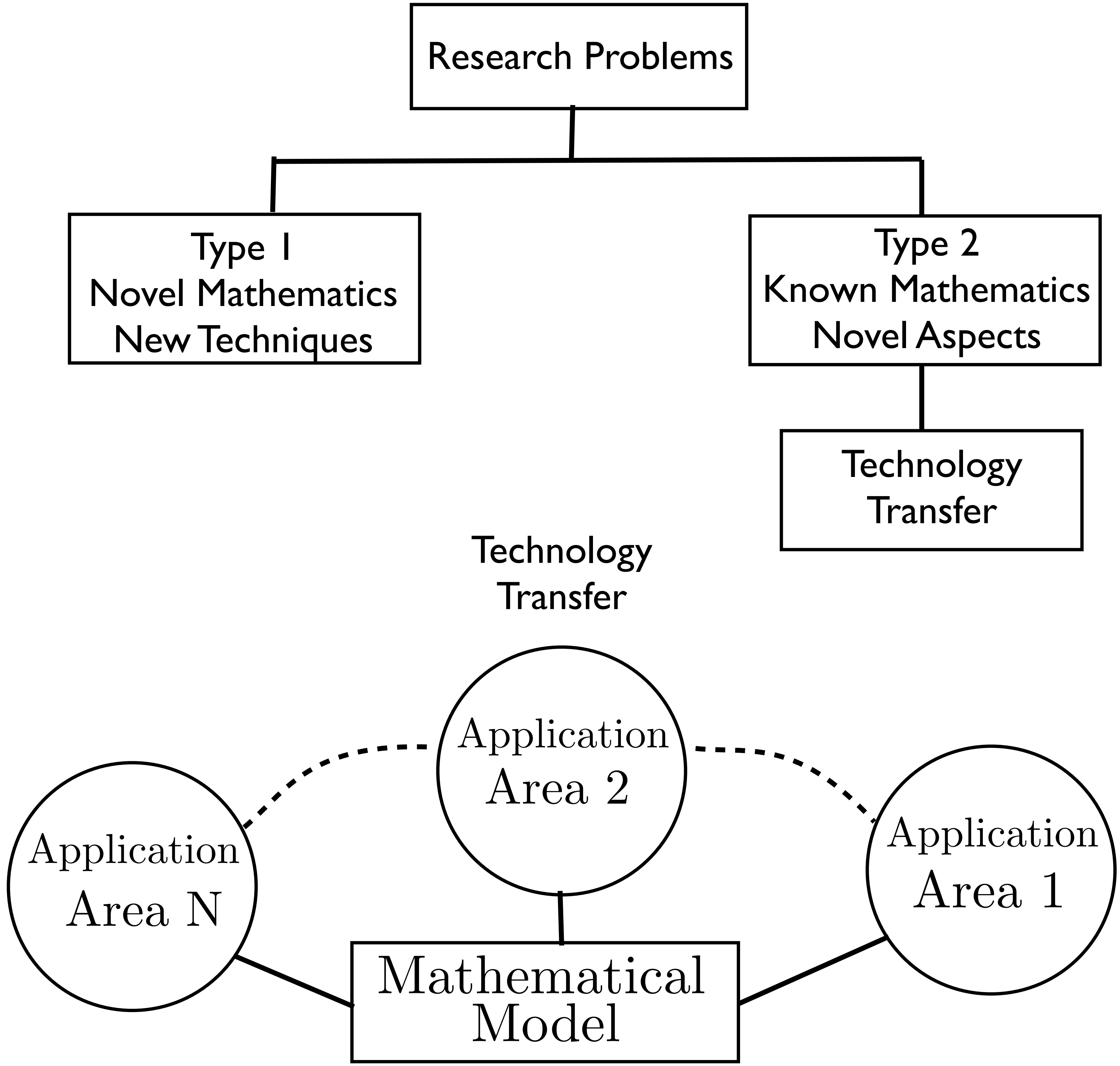
Modelling Process: Overview



Modelling Process: Analysis



Modelling Process: Research Problems



Neutron Transport - Introduction

- Neutron Motion - Boltzmann Transport Equation
  - Linear Integro-differential Equation
- Diffusion Theory - Common Approximation
  - Neutron Flux = Fick's law
  - Good working results
- Objective: Derive Diffusion Equation + B.C.s
- References:

G.J. Habetler & B.J. Matkowsky, Uniform asymptotic expansions in tranport theory with small mean free paths and the diffusion approximation, J. Math. Phys. 16 (1975) 846-854.

K.M. Case and P.F. Zweifel, Linear Transport Theory, Addison-Wesley, (1967)

Neutron Transport - Problem Formulation

- Neutron conservation statement: $$\frac{\partial N}{\partial t} + \nabla \cdot \mathbf{H} = q$$

- $N(\mathbf{r}, \boldsymbol{\Omega}, t)$ (angular) neutron density (number per unit volume)

- Flux $\mathbf{H} = N\mathbf{v}$ $\quad \mathbf{v} = v\boldsymbol{\Omega}$ $\quad v$ speed (constant)

  $\nabla \cdot \mathbf{H} = v\boldsymbol{\Omega} \cdot \nabla N$ $\quad \boldsymbol{\Omega}$ direction

- Source
$$q = -v\sigma(\mathbf{r})\, N(\mathbf{r}, \boldsymbol{\Omega}, t) + \frac{1}{4\pi}\int_{\mathbb{S}^2} v\sigma_s(\mathbf{r}, \boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')\, N(\mathbf{r}, \boldsymbol{\Omega}', t)\, dS(\boldsymbol{\Omega}') + \frac{\nu(\mathbf{r})}{4\pi} v\sigma_f(\mathbf{r}) \int_{\mathbb{S}^2} N(\mathbf{r}, \boldsymbol{\Omega}', t)\, dS(\boldsymbol{\Omega}') + Q(\mathbf{r}, t)$$

Total macroscopic cross-section

$$\sigma(\mathbf{r}) = \sigma_f(\mathbf{r}) + \sigma_c(\mathbf{r}) + \frac{1}{4\pi}\int_{\mathbb{S}^2} \sigma_s(\mathbf{r}, \boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')\, dS(\boldsymbol{\Omega}')$$

Neutron Transport - Problem Formulation

- Isotropic scattering $\sigma_s(\mathbf{r}, \mathbf{\Omega}.\mathbf{\Omega}') = \sigma_s(\mathbf{r})$

- $$\frac{1}{v}\frac{\partial \psi(\mathbf{r}, \mathbf{\Omega}, t)}{\partial t} + \mathbf{\Omega} \cdot \nabla \psi(\mathbf{r}, \mathbf{\Omega}, t) + \sigma(\mathbf{r})\psi(\mathbf{r}, \mathbf{\Omega}, t) = \frac{\sigma(\mathbf{r})c(\mathbf{r})}{4\pi} \int_{\mathbb{S}^2} \psi(\mathbf{r}, \mathbf{\Omega}', t)\, dS(\mathbf{\Omega}') + Q(\mathbf{r}, t)$$

- $c(\mathbf{r}) = \dfrac{\sigma_s(\mathbf{r}) + \nu \sigma_f(\mathbf{r})}{\sigma(\mathbf{r})}$ mean number secondary neutrons

Neutron Transport - Problem Formulation

- 1-D problem
- slab geometry

  1 space coordinate x

  1 direction coordinate $\mu = \mathbf{\Omega} \cdot \mathbf{i} = \cos\theta$

  $\theta \in [0, \pi]$

- $\psi = \psi(x, \mu, t)$

$$\frac{1}{v}\frac{\partial \psi(x,\mu,t)}{\partial t} + \mu \frac{\partial \psi(x,\mu,t)}{\partial x} + \sigma(x)\psi(x,\mu,t) = \frac{\sigma(x)c(x)}{2}\int_{-1}^{1} \psi(x,\mu',t)\, d\mu' + Q(x,t)$$

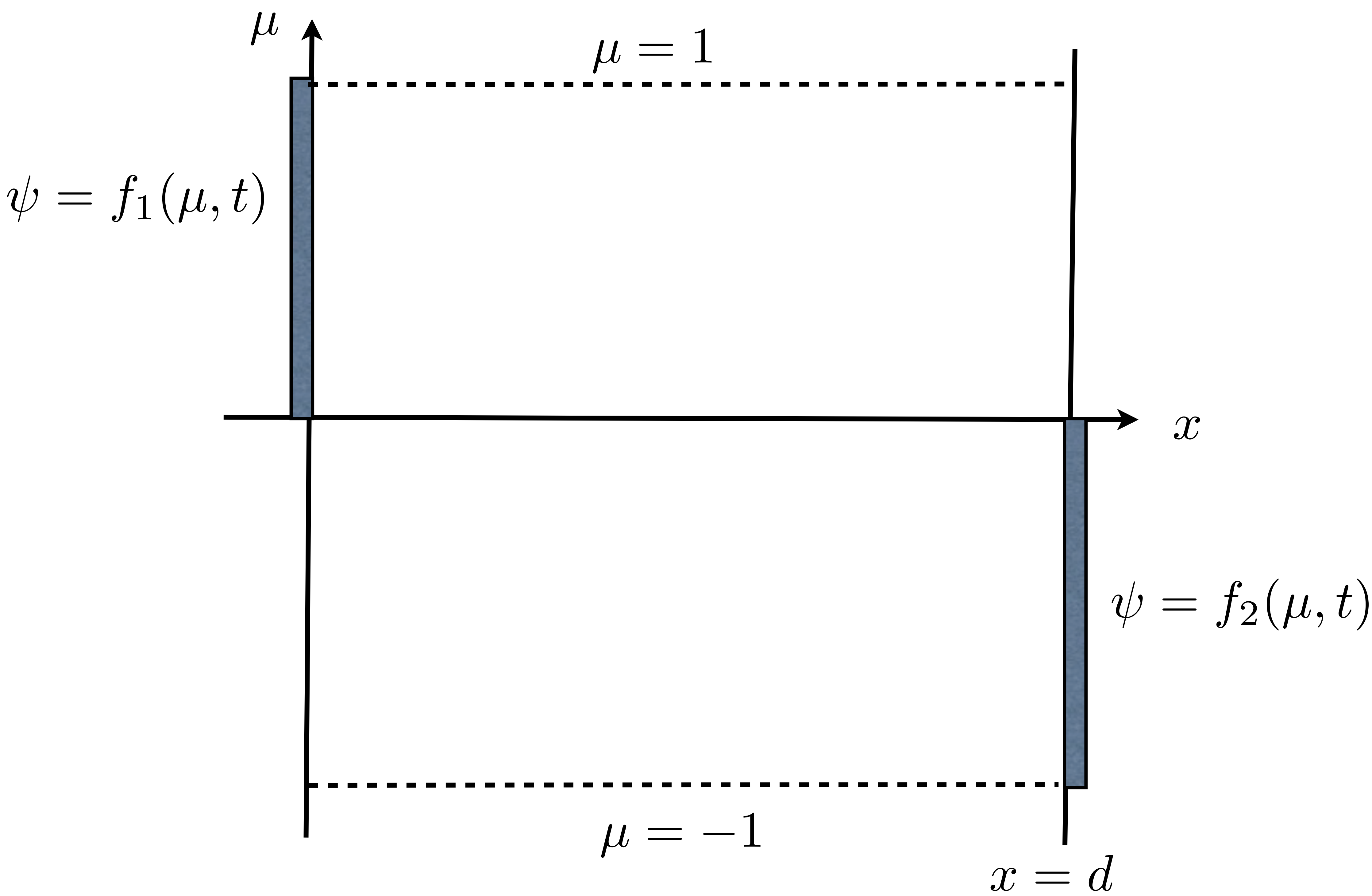
$$0 < x < d \quad -1 \leq \mu \leq 1$$

$$x = 0 \qquad \psi = f_1(\mu, t) \qquad \mu > 0$$

$$x = d \qquad \psi = f_2(\mu, t) \qquad \mu < 0$$

at $t = 0$ $\quad \psi = g(x, \mu)$ $\quad$ for $0 \leq x \leq d, -1 \leq \mu \leq 1$

Neutron Transport - Problem Formulation



Neutron Transport - Non-dimensionalisation

- Non-dimensionalise:

- $x = d\bar{x} \qquad t = \frac{d}{v}\bar{t} \qquad \psi = \psi_0\bar{\psi} \qquad Q = \psi_0\sigma(x)\bar{Q}$

  $f_1 = \psi_0\bar{f}_1 \qquad f_2 = \psi_0\bar{f}_2 \qquad g = \psi_0\bar{g}$

- $\sigma = \sigma_0\bar{\sigma}(\bar{x}) \qquad c = c(\bar{x})$

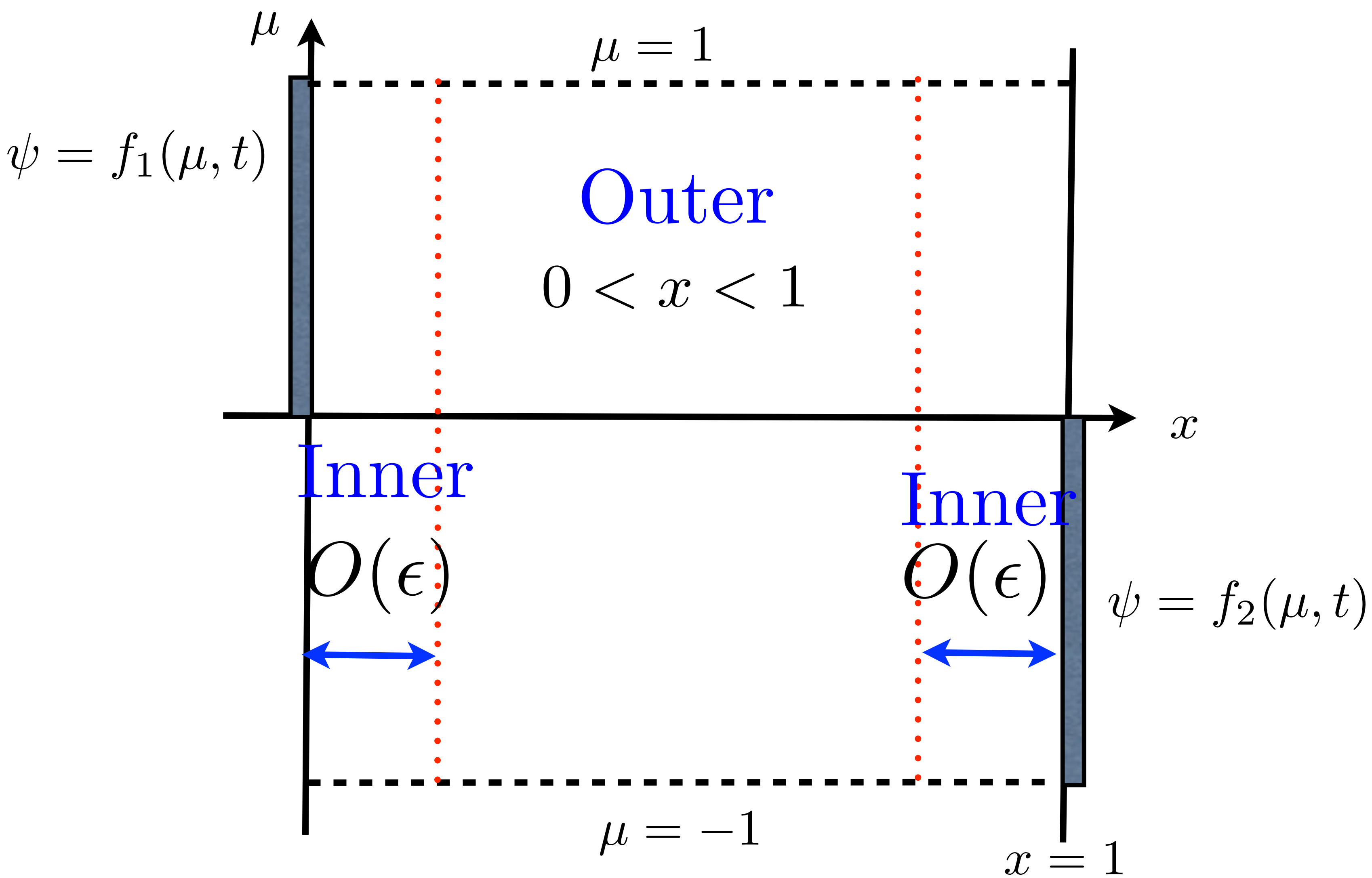
Neutron Transport - Non-dimensionalisation

- Dimensionless Problem:
- $$\frac{\epsilon}{\bar\sigma(\bar x)}\frac{\partial\bar\psi}{\partial\bar t}+\frac{\epsilon\mu}{\bar\sigma(\bar x)}\frac{\partial\bar\psi(\bar x,\mu,\bar t)}{\partial\bar x}+\bar\psi(\bar x,\mu,\bar t) = \frac{c(\bar x)}{2}\int_{-1}^{1}\bar\psi(\bar x,\mu',\bar t)\,d\mu'+\bar Q(\bar x,\bar t)$$
  $$0<\bar x<1,\ -1\le\mu\le 1,$$
  on $\bar x = 0$ $\quad \bar\psi=\bar f_1(\mu,\bar t) \quad \mu>0$
  on $\bar x = 1$ $\quad \bar\psi=\bar f_2(\mu,\bar t) \quad \mu<0$
  at $\bar t = 0$ $\quad \bar\psi=\bar g(\bar x,\mu) \quad$ for $0\le\bar x\le 1,\ -1\le\mu\le 1$
- Dimensionless Parameter $\quad \epsilon=\dfrac{1}{\sigma_0 d} \sim 10^{-9}-10^{-3}$
  Mean free path
- 4 dimensional parameters (d, v, ψ_0, σ_0) $\longrightarrow$ 1 dimensionless parameter ϵ

Neutron Transport - Matched Asymptotics

- Matched Asymptotic Expansions $\quad \epsilon \to 0$

- $$\frac{\epsilon}{\sigma(x)}\frac{\partial \psi}{\partial t} + \frac{\epsilon\mu}{\sigma(x)}\frac{\partial \psi(x,\mu,t)}{\partial x} + \psi(x,\mu,t) = \frac{c(x)}{2}\int_{-1}^{1}\psi(x,\mu',t)\,d\mu' + Q(x,t)$$



Neutron Transport - Outer Expansion

- Outer region $0 < x < 1$

- Pose

$$\psi = \psi_0(x,\mu,t) + \epsilon\psi_1(x,\mu,t) + \epsilon^2\psi_2(x,\mu,t) + \ldots$$

as $\epsilon \to 0$

- $c(x) = c_0(x) + \epsilon c_1(x) + \epsilon^2 c_2(x) + \ldots$

$$Q(x,t) = Q_0(x,t) + \epsilon Q_1(x,t) + \epsilon^2 Q_2(x,t) + \ldots$$

- $t = \dfrac{\epsilon}{\delta}\tau \qquad \delta = K_0 + K_1\epsilon + K_2\epsilon^2 + \ldots$

Neutron Transport - Outer Expansion

- At $O(\epsilon^0)$:

- $$\frac{K_0}{\sigma(x)}\frac{\partial \psi_0}{\partial \tau} + \psi_0 = \frac{c_0(x)}{2}\int_{-1}^{1} \psi_0(x,\mu',t)d\mu' + Q_0(x,t)$$

- $\psi_0 = \psi_0(x,t)$ $$\frac{K_0}{\sigma(x)}\frac{\partial \psi_0}{\partial \tau} + (1 - c_0(x))\,\psi_0 = Q_0(x,\tau)$$

$K_0 = 0$ $c_0 = 1$ $Q_0 = 0$

Neutron Transport - Outer Expansion

- At $O(\epsilon)$:
- $$\frac{K_1}{\sigma(x)}\frac{\partial \psi_0}{\partial \tau} + \frac{\mu}{\sigma(x)}\frac{\partial \psi_0}{\partial x} + \psi_1 = \frac{c_0(x)}{2}\int_{-1}^{1} \psi_1(x,\mu',\tau)d\mu' + \frac{c_1(x)}{2}\int_{-1}^{1} \psi_0(x,\tau)d\mu' + Q_1(x,\tau)$$
- $\psi_1 = \psi_{10}(x,\tau) + \mu\psi_{11}(x,\tau)$
  $$\frac{K_1}{\sigma(x)}\frac{\partial \psi_0}{\partial \tau} = c_1(x)\psi_0 + Q_1(x,\tau)$$
  $$\frac{1}{\sigma(x)}\frac{\partial \psi_0}{\partial x} + \psi_{11} = 0$$
- $K_1 = 0 \qquad c_1 = 0 \qquad Q_1 = 0$

Neutron Transport - Outer Expansion

- At $O(\epsilon^2)$: $$\frac{K_2}{\sigma(x)}\frac{\partial \psi_0}{\partial \tau} + \frac{\mu}{\sigma(x)}\frac{\partial \psi_1}{\partial x} + \psi_2 = \frac{1}{2}\int_{-1}^{1}\psi_2(x,\mu',\tau)d\mu' + \frac{c_2(x)}{2}\int_{-1}^{1}\psi_0(x,\tau)d\mu' + Q_2(x,\tau)$$

- $$\psi_2 = \psi_{20}(x,\tau) + \mu\psi_{21}(x,\tau) + \mu^2\psi_{22}(x,\tau)$$
$$\frac{K_2}{\sigma(x)}\frac{\partial \psi_0}{\partial \tau} = \frac{1}{3}\psi_{22} + c_2(x)\psi_0 + Q_2(x,\tau)$$
$$\frac{1}{\sigma(x)}\frac{\partial \psi_{10}}{\partial x} + \psi_{21} = 0$$
$$\frac{1}{\sigma(x)}\frac{\partial \psi_{11}}{\partial x} + \psi_{22} = 0$$

- $$\frac{K_2}{\sigma(x)}\frac{\partial \psi_0}{\partial \tau} = \frac{1}{3\sigma(x)}\frac{\partial}{\partial x}\left(\frac{1}{\sigma(x)}\frac{\partial \psi_0}{\partial x}\right) + c_2(x)\psi_0 + Q_2$$

- $t = \frac{\tau}{\epsilon}$ $\quad c(x) = 1 + \epsilon^2 c_2(x)$ $\quad Q(x,t) = \epsilon^2 Q_2(x,\tau)$

Neutron Transport - Inner Expansion

- Inner region at $x = 0$
- $x = \epsilon y \qquad \psi = \Psi$
- $\sigma(\epsilon y) = \sigma(0) \quad Q(\epsilon y, t) = \epsilon^2 Q_2(0, \tau)$
- $$\frac{\epsilon^2}{\sigma(0)} \frac{\partial \Psi}{\partial \tau} + \frac{\mu}{\sigma(0)} \frac{\partial \Psi(y, \mu, \tau)}{\partial y} + \Psi(y, \mu, \tau)$$
  $$\frac{c(\epsilon y)}{2} \int_{-1}^{1} \Psi(y, \mu', \tau)\, d\mu' \; + \epsilon^2 Q_2(0, \tau)$$
- $c(\epsilon y) = 1 + \epsilon^2 c_2(0) + O(\epsilon^3)$
- Domain: $0 < y < \infty, -1 \leq \mu \leq 1$

Neutron Transport - Inner Expansion

- Pose: $\Psi = \Psi_0(y,\mu,\tau) + \epsilon\Psi_1(y,\mu,\tau) + \epsilon^2\Psi_2(y,\mu,\tau) + \ldots$

  as $\epsilon \to 0$

- At $O(\epsilon^0)$:

$$\frac{\mu}{\sigma(0)}\frac{\partial\Psi_0}{\partial y} + \Psi_0 = \frac{1}{2}\int_{-1}^{1}\Psi_0(y,\mu',\tau)d\mu'$$

$$\Psi_0(0,\mu,\tau) = f_1(\mu,\tau)$$

- General solution:

$$\Psi_0(y,\mu,\tau) = a_0(\tau) + b_0(\tau)(\sigma(0)y - \mu) + \int_{-1}^{1} A_0(\nu,\tau)\phi_\nu(\mu)e^{-y\sigma(0)/\nu}\,d\nu$$

- $\phi_\nu(\mu) = \frac{\nu}{2}P\frac{1}{\nu-\mu} + \lambda(\nu)\delta(\nu-\mu)$ $\qquad \lambda(\nu) = 1 - \nu\tanh^{-1}\nu$
- $A_0(\nu,\tau) = 0 \qquad \nu < 0$

Neutron Transport - Inner Expansion

- Orthogonality Conditions:

- $$\int_0^1 \phi_\nu(\mu)\,\gamma(\mu)\,d\mu = 0$$

- $$\int_0^1 \phi_\nu(\mu)\,\phi_{\nu'}(\mu)\,\gamma(\mu)\,d\mu = \frac{\gamma(\nu)}{\nu} N(\nu)\delta(\nu - \nu')$$

- $$\gamma(\mu) = \frac{3\mu}{2X(-\mu)} \qquad N(\nu) = \nu\left(\lambda(\nu)^2 + \frac{\pi^2\nu^2}{4}\right)$$

- $$X(z) = \frac{1}{1-z}\exp\left(\frac{1}{\pi}\int_0^1 \frac{1}{(\mu' - z)}\tan^{-1}\left[\frac{\pi\mu'}{2\lambda(\mu')}\right]\right) d\mu'$$

Neutron Transport - Inner Expansion

- $$a_0(\tau) = \frac{\gamma_1}{\gamma_0} b_0(\tau) + \frac{1}{\gamma_0} \int_0^1 f_1(\mu, \tau)\gamma(\mu) d\mu$$

- $$A_0(\nu, \tau) = -\frac{\nu^2 \gamma_0 b_0(\tau)}{2\gamma(\nu)N(\nu)} + \frac{\nu}{\gamma(\nu)N(\nu)} \int_0^1 f_1(\mu, \tau)\, \phi_\nu(\mu)\, \gamma(\mu)\, d\mu$$

$$\gamma_i = \int_0^1 \mu^i \gamma(\mu) d\mu \qquad \text{for } i = 0, 1$$

- $b_0(\tau)$ determined by matching to outer

Neutron Transport - Matching

- Overlap domain: $\epsilon \ll x = \epsilon y \ll 1$
- Outer in Inner variables:

$$\Psi = \psi_0(\epsilon y, \mu, \tau) + \epsilon \psi_1(\epsilon y, \mu, \tau) + \epsilon^2 \psi_2(\epsilon y, \mu, \tau) + \ldots$$

$$= \psi_0(0, \tau) + \epsilon \left[y \frac{\partial \psi_0(0, \mu, \tau)}{\partial x} + \psi_1(0, \mu, \tau) \right]$$

$$+\epsilon^2 \left[\frac{y^2}{2} \frac{\partial^2 \psi_0(0, \mu, \tau)}{\partial x^2} + y \frac{\partial \psi_1(0, \mu, \tau)}{\partial x} + \psi_2(0, \mu, \tau) \right] + \ldots$$

- Outer limit of Inner:
- $\Psi = \Psi_0(y, \mu, \tau) + \epsilon \Psi_1(y, \mu, \tau) + \epsilon^2 \Psi_2(y, \mu, \tau) + \ldots$

  as $y \to \infty$
- At $O(\epsilon^0)$: $\lim_{y \to \infty} \Psi_0(y, \mu, \tau) = \psi_0(0, \tau)$

Neutron Transport - Matching

- $$\Psi_0(y,\mu,\tau) = b_0(\tau)\sigma(0)y + (a_0(\tau) - b_0(\tau)\mu) + o(1)$$
  $$\text{as } y \to \infty$$

- $b_0(\tau) = 0 \qquad a_0(\tau) = \psi_0(0,\tau)$

Neutron Transport - Summary

- Governing Equation:

$$\frac{K_2}{\sigma(x)}\frac{\partial \psi_0}{\partial \tau} = \frac{1}{3\sigma(x)}\frac{\partial}{\partial x}\left(\frac{1}{\sigma(x)}\frac{\partial \psi_0}{\partial x}\right) + c_2(x)\psi_0 + Q_2$$

$$\text{in } 0 < x < 1, \tau > 0$$

- Boundary Conditions:

$$\psi_0(0,\tau) = \int_0^1 f_1(\mu,\tau)\gamma(\mu)d\mu$$

$$\psi_0(1,\tau) = \int_0^1 f_2(\mu,\tau)\gamma(\mu)d\mu$$

- Initial Condition: $\psi_0(x,0) = G(x)$??

The End

Thank You !