Proof:

1. Follows from linearity of integration

2. Integration by parts gives

$$\int_0^\infty f'(t)\, e^{-st}\, dt = f(t) e^{-st}\Big|_0^\infty + s\int_0^\infty f(t) e^{-st}\, dt$$

$$\boxed{\begin{array}{l} u = e^{-st};\ du = -s e^{-st} \\ dv = f';\ v = f \end{array}}$$

Since f is of exponential order, $\lim_{t\to\infty} f(t) e^{-st} = 0 \quad \forall s \in \mathbb{C}$ with $Re(s)$ sufficiently large.

$$\therefore \mathcal{L}\{f'(t)\}(s) = -f(0) + s\,\mathcal{L}\{f(t)\}(s)$$

3. Let $F(t) = \int_0^t f(\tau)\, d\tau$ and thus $F(0)=0$, $F'(t)=f(t)$

By 2: $\quad \mathcal{L}\{F'(t)\}(s) = s\,\mathcal{L}\{F(t)\}(s) - F(0)^{=0}$

$$\therefore \mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}(s) = \frac{1}{s}\hat{f}(s) .$$

4. For any $a \in \mathbb{C}$,

$$\mathcal{L}\{e^{-at} f(t)\}(s) = \int_0^\infty e^{-at} f(t) e^{-st} dt$$

$$= \int_0^\infty f(t) e^{-(a+s)t} dt = \hat{f}(a+s).$$

5. $$\mathcal{L}\{f(t-T)H(t-T)\}(s) = \int_{t=0}^\infty f(t-T)H(t-T)e^{-st} dt$$

$$= \int_{t=T}^\infty f(t-T) e^{-st} dt$$

$\tau = t - T$
$d\tau = dt$

$$= \int_{\tau=0}^\infty f(\tau) e^{-s(\tau+T)} d\tau$$

$$= e^{-sT} \int_{\tau=0}^\infty f(\tau) e^{-s\tau} d\tau = e^{-sT} \hat{f}(s).$$

Examples:

1. For $a \in \mathbb{R}$: $\mathcal{L}\{\cos at\}(s) = \mathcal{L}\left\{\frac{1}{2}e^{iat} + \frac{1}{2}e^{-iat}\right\}(s)$

$$= \frac{1}{2}\mathcal{L}\{e^{iat}\}(s) + \frac{1}{2}\mathcal{L}\{e^{-iat}\}(s) \quad \text{(by linearity)}$$

$$= \frac{1}{2}\cdot\frac{1}{(s-ia)} + \frac{1}{2}\cdot\frac{1}{(s+ia)} = \frac{s}{s^2+a^2} \qquad \operatorname{Re}(s) > 0 \text{ using (2.2)}$$

2. By transform of a derivative:

$$\mathcal{L}\{\sin(at)\}(s) = s\,\mathcal{L}\left\{-\frac{1}{a}\cos(at)\right\}(s) + \frac{1}{a}$$

$$= \frac{1}{a} - \frac{s}{a}\mathcal{L}\{\cos(at)\}(s) = \frac{1}{a} - \frac{s^2}{a(s^2+a^2)}$$

$$= \frac{a}{s^2+a^2} \qquad \text{for } \operatorname{Re}(s) > 0.$$

⌈Or proceeding as in 1.⌋

3. By transform of an integral: For $n \in \mathbb{N}$,

$$\mathcal{L}\{t^n\}(s) = \mathcal{L}\left\{ n\int_0^t \tau^{n-1}\,d\tau \right\}(s)$$

$$= \frac{n}{s}\mathcal{L}\{t^{n-1}\}(s) \qquad \text{by } (2.4)$$

$$= \frac{n(n-1)}{s^2}\mathcal{L}\{t^{n-2}\}(s)$$

$$= \frac{n(n-1)\ldots 1}{s^n}\mathcal{L}\{t^0\}(s) = \frac{n!}{s^{n+1}} \qquad \mathrm{Re}(s) > 0$$

⌈Or: from definition $\mathcal{L}\{t^n\}(s) = \int_0^\infty t^n e^{-st}\,dt$

use integration by-parts

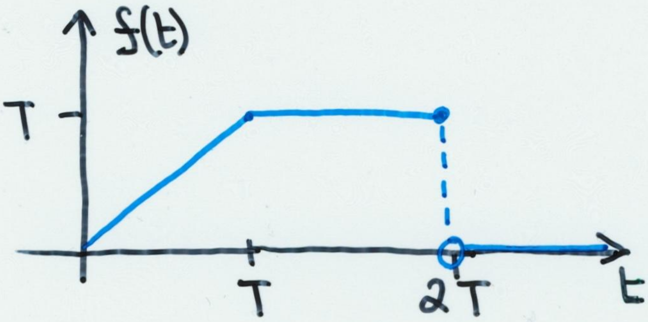
differentiate (t^n) integrate (e^{-st}) ⌋

4. By the damping formula:

$$\mathcal{L}\{e^{-\lambda t}\cos(at)\}(s) = \mathcal{L}\{\cos(at)\}(s+\lambda)$$

$$= \frac{s+\lambda}{(s+\lambda)^2 + a^2}$$

5. Consider $f(t) = \begin{cases} t, & 0 < t \leq T \\ T, & T < t \leq 2T \\ 0, & t > 2T. \end{cases}$



piecewise continuous function.

Now $f(t) = t\left(1 - H(t-T)\right) + T\left(H(t-T) - H(t-2T)\right)$

$= t - (t-T)H(t-T) - T\,H(t-2T)$

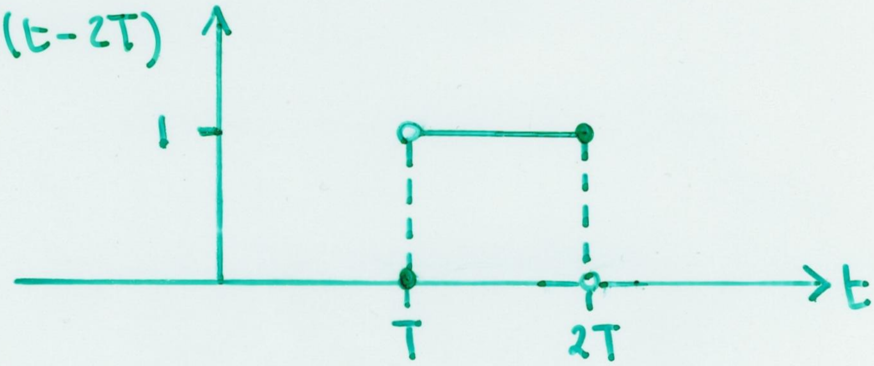
Thus

$\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{t\}(s) - \mathcal{L}\{(t-T)H(t-T)\}(s)$

$\qquad - T\,\mathcal{L}\{H(t-2T)\}(s)$

$= \dfrac{1}{s^2} - \dfrac{e^{-sT}}{s^2} - \dfrac{T}{s}e^{-2sT}$ by delay formula.

<u>NB</u> $H(t-T) - H(t-2T)$



<u>NB</u> Directly: $\hat{f}(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^T t e^{-st}\,dt + \int_T^{2T} T e^{-st}\,dt$

same result.

2.2. Solving ODEs with LT.

The Laplace Transform turns an ODE into an algebraic equation, whose solutions can then be transformed back (inverted) to solutions of the ODE.

Examples

1. Consider the second order IVP

$$x''(t) - 3x'(t) + 2x(t) = 1, \quad x(0) = x'(0) = 0$$

Take the LT of the ODE:

$$\mathcal{L}\{x''(t) - 3x'(t) + 2x(t)\}(s) = \mathcal{L}\{1\}(s)$$

$$\therefore \mathcal{L}\{x''(t)\}(s) - 3\mathcal{L}\{x'(t)\}(s) + 2\mathcal{L}\{x(t)\}(s) = \frac{1}{s}$$

by linearity.

$$\therefore (s^2\hat{x} - s\underset{0}{\underset{\shortparallel}{x(0)}} - \underset{0}{\underset{\shortparallel}{x'(0)}}) - 3(s\hat{x} - \underset{0}{\underset{\shortparallel}{x(0)}}) + 2\hat{x} = \frac{1}{s}$$

$$\therefore \hat{x}(s^2 - 3s + 2) = \frac{1}{s} \Rightarrow \hat{x}(s) = \frac{1}{s(s-1)(s-2)}$$

Use partial fractions: $\hat{x}(s) = \frac{1/2}{s} - \frac{1}{(s-1)} + \frac{1/2}{(s-2)}$

Invert: $x(t) = \frac{1}{2} - e^{t} + \frac{1}{2}e^{2t}$ $(*)$

using standard transforms.

Notes:

1. Inverting $\hat{x}(s)$ to give $x(t)$ requires the use of known transforms i.e. we ask what $f(t)$ gives the corresponding $\hat{f}(s)$.

2. Always check that the answer $(*)$ satisfies the original IVP i.e. ODE + I.C.