2. Solve the IVP:

$$x''(t) + 4x(t) = \cos 3t\ , \quad x(0)=1,\ x'(0)=-3.$$

Take LT of ODE:

$$s^2\hat{x} - s\underbrace{x(0)}_{=1} - \underbrace{x'(0)}_{=-3} + 4\hat{x} = \frac{s}{(s^2+9)}$$

$$\therefore\ \hat{x}(s^2+4) = \frac{s}{(s^2+9)} + s - 3$$

$$\therefore\ \hat{x}(s) = \frac{s}{(s^2+4)(s^2+9)} + \frac{s-3}{(s^2+4)}$$

$$= \frac{1}{5}\left(\frac{s}{(s^2+4)} - \frac{s}{(s^2+9)}\right) + \frac{s-3}{(s^2+4)}$$

$$= \frac{6}{5}\cdot\frac{s}{(s^2+4)} - \frac{3}{(s^2+4)} - \frac{s}{5(s^2+9)}$$

Inverting using standard transforms:

$$x(t) = \frac{6}{5}\cos 2t - \frac{3}{2}\sin 2t - \frac{1}{5}\cos 3t\ .$$

The L.T. is also useful for solving systems of linear ODEs:

3. Consider $\dot{x}_1(t) - 2x_2(t) = 4t$, $\quad x_1(0) = 4$;

$\dot{x}_2(t) + 2x_2(t) - 4x_1(t) = -4t - 2$, $\quad x_2(0) = -5$.

Take LT: $s\hat{x}_1 - \underbrace{x_1(0)}_{=4} - 2\hat{x}_2 = \dfrac{4}{s^2}$,

$$s\hat{x}_2 - \underbrace{x_2(0)}_{=-5} + 2\hat{x}_2 - 4\hat{x}_1 = -\frac{4}{s^2} - \frac{2}{s}.$$

$$\therefore \quad s\hat{x}_1 - 2\hat{x}_2 = \frac{4 + 4s^2}{s^2} \qquad (a)$$

$$-4\hat{x}_1 + (2+s)\hat{x}_2 = -\frac{(4 + 2s + 5s^2)}{s^2} \qquad (b)$$

Solving for $\hat{x}_1$: $(2+s)(a) + 2(b)$:

$$((2+s)s - 8)\hat{x}_1 = \frac{(4s^2+4)(2+s)}{s^2} - \frac{(10s^2 + 4s + 8)}{s^2}$$

$$\therefore (s+4)(s-2)\hat{x}_1 = \frac{4s^3 - 2s^2}{s^2} = 4s - 2$$

$$\therefore \quad \hat{x}_1(s) = \frac{4s-2}{(s-2)(s+4)} = \frac{1}{(s-2)} + \frac{3}{(s+4)}$$

Inverting (using standard transforms):

$$x_1(t) = e^{2t} + 3e^{-4t}$$

and thus $x_2(t) = \frac{1}{2}\dot{x}_1(t) - 2t = e^{2t} - 6e^{-4t} - 2t$.

$\square$

Notation:

If $\mathcal{L}\{f(t)\}(s) = \hat{f}(s)$ is the L.T. of $f(t)$

then $\mathcal{L}^{-1}\{\hat{f}(s)\}(t) = f(t)$ is the inverse Laplace Transform.

Note: sometimes written as

$$\mathcal{L}\{f(t)\} = \hat{f}(s) \quad \text{and} \quad \mathcal{L}^{-1}\{\hat{f}(s)\} = f(t).$$

2.3. The convolution integral.

<u>Definition:</u> The function $(f*g)(t) := \int_{\tau=0}^{t} f(t-\tau)g(\tau)\,d\tau$

is called the convolution of f and g. $\square$

What is its LT ?

$$\mathcal{L}\{(f*g)(t)\}(s) = \int_{t=0}^{\infty} \left[\int_{\tau=0}^{t} f(t-\tau)g(\tau)\,d\tau \right] e^{-st}\,dt$$

$$= \int_{t=0}^{\infty} \left[\int_{\tau=0}^{\infty} H(t-\tau)\, f(t-\tau)\, g(\tau)\,d\tau \right] e^{-st}\,dt$$

$H(t-\tau) = \begin{cases} 1 & t>\tau \\ 0 & t<\tau \end{cases}$

$$= \int_{\tau=0}^{\infty} \left[\underbrace{\int_{t=0}^{\infty} H(t-\tau) f(t-\tau)\, e^{-st}\,dt}_{\mathcal{L}\{f(t-\tau)H(t-\tau)\}(s)} \right] g(\tau)\,d\tau$$

$$= \int_{\tau=0}^{\infty} \hat{f}(s)\, e^{-s\tau}\, g(\tau)\,d\tau$$ by the Delay formula.

$$= \hat{f}(s) \int_{\tau=0}^{\infty} g(\tau)\, e^{-s\tau}\,d\tau = \hat{f}(s)\,\hat{g}(s).$$

Thus $\mathcal{L}\{(f*g)(t)\}(s) = \hat{f}(s)\,\hat{g}(s)$

or $\mathcal{L}^{-1}\{\hat{f}(s)\,\hat{g}(s)\}(t) = (f*g)(t)$.

Note: Convolution is commutative:

$$(f*g)(t) = \int_{\tau=0}^{t} f(t-\tau)\,g(\tau)\,d\tau = -\int_{\sigma=t}^{0} f(\sigma)\,g(t-\sigma)\,d\sigma$$

$$\boxed{\begin{aligned}\sigma &= t-\tau\\ d\sigma &= -d\tau\end{aligned}} \qquad = \int_{\sigma=0}^{t} g(t-\sigma)\,f(\sigma)\,d\sigma = (g*f)(t).$$

Examples:

1. Compute $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\}$

Now, $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\overbrace{\dfrac{1}{(s^2+1)}}^{=\hat{f}(s)} \cdot \overbrace{\dfrac{s}{(s^2+1)}}^{=\hat{g}(s)}\right\}$

$= \sin t * \cos t$ since $\mathcal{L}^{-1}\{\hat{f}(s)\} = \sin t$

$\mathcal{L}^{-1}\{\hat{g}(s)\} = \cos t$

$$= \int_{\tau=0}^{t} \sin(t-\tau)\cos\tau\,d\tau$$

$$= \int_{\tau=0}^{t} (\sin t\cos\tau - \cos t\sin\tau)\cos\tau\,d\tau$$

$$= \sin t \int_{\tau=0}^{t} \cos^2\tau\,d\tau - \cos t \int_{\tau=0}^{t} \sin\tau\cos\tau\,d\tau$$

$$= \sin t \int_{\tau=0}^{t} \tfrac{1}{2}(1+\cos 2\tau)\,d\tau - \cos t \int_{\tau=0}^{t} \tfrac{1}{2}\sin 2\tau\,d\tau$$

$$= \frac{1}{2}\sin t\left[\tau + \frac{1}{2}\sin 2\tau\right]_{\tau=0}^{t} - \frac{1}{2}\cos t\left[-\frac{1}{2}\cos 2\tau\right]_{\tau=0}^{t}$$

$$= \frac{1}{2}\sin t\left(t + \frac{1}{2}\sin 2t\right) - \frac{1}{2}\cos t\left(\frac{1}{2} - \frac{1}{2}\cos 2t\right)$$

$$= \frac{1}{2}t\sin t + \frac{1}{4}\sin t\sin 2t - \frac{1}{4}\cos t + \frac{1}{4}\cos t\cos 2t$$

$$= \frac{1}{2}t\sin t + \frac{1}{4}\cos(2t-t) - \frac{1}{4}\cos t$$

$$= \frac{1}{2}t\sin t$$

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2. Solve the IVP:

$$\ddot{y}(t) + y(t) = f(t), \quad y(0)=0, \ \dot{y}(0)=1$$

where $f(t) = \begin{cases} 1, & t \in [0,1] \\ 0, & t > 1. \end{cases}$

Answer:

(a) Write $f(t) = H(t) - H(t-1)$.

(b) Take LT of ODE:

$$s^2\hat{y} - sy(0) - \dot{y}(0) + \hat{y} = \mathcal{L}\{H(t)\} - \mathcal{L}\{H(t-1)\}$$

$$\therefore (s^2+1)\hat{y} - 1 = \frac{1}{s} - \frac{e^{-s}}{s}$$