1.2 Autonomous Homogeneous Systems

(A constant matrix, $g \equiv 0$)

Theorem (Existence & Uniqueness of IVPs):

Let $A \in \mathbb{C}^{n \times n}$, $x_0 \in \mathbb{C}^n$. Then the IVP

$$\dot{x}(t) = A x(t) \quad , \quad x(t_0) = x_0 \qquad (1.4)$$

has a unique solution.

Proof (sketch):

a) Existence: Define the matrix exponential for $Y \in \mathbb{C}^{n \times n}$

$$\exp(tY) = \sum_{k=0}^{\infty} \frac{1}{k!}(tY)^k \equiv I + tY + \frac{t^2}{2!}Y^2 + \dots$$

Property 1: $\exp(tY)$ converges $\forall\, t \in \mathbb{R}$

Property 2: $\frac{d}{dt}\exp(tY) = Y\exp(tY) = \exp(tY)\,Y$

Define $x(t) = \exp\big((t-t_0)A\big)\,x_0$

Then $x(t_0) = \exp(0.A)\,x_0 = I x_0 = x_0$

and $\dot{x}(t) = A\exp((t-t_0)A)\,x_0 = A x(t)$.

Hence $x(t)$ solves the IVP (1.4).

Note: It is difficult to compute the matrix exponential $\exp(tA)$ explicitly (discussed later).

b) Uniqueness: Let $y(t)$ be another solution of (1.4)

Consider $h(t) = x(t) - y(t)$

Then $\dot{h}(t) = \dot{x}(t) - \dot{y}(t) = Ax(t) - Ay(t) = Ah(t)$

$\therefore\ \dot{h}(t) = Ah(t)$ and $h(t_0) = 0$.

We wish to show $h(t) \equiv 0$. Suffice to show $\|h(t)\| = 0$.

For $h(t) = \begin{pmatrix} h_1(t) \\ \vdots \\ h_n(t) \end{pmatrix} \in \mathbb{C}^n$, $\|h\| := (|h_1|^2 + \cdots + |h_n|^2)^{1/2}$

Now, $\frac{dh}{ds}(s) = Ah(s) \Rightarrow \int_{t_0}^{t} \frac{dh}{ds}\, ds = \int_{t_0}^{t} Ah(s)\, ds$

i.e. $h(t) - \underset{\overset{\shortparallel}{0}}{h(t_0)} = \int_{t_0}^{t} Ah(s)\, ds$

$$\therefore\ 0 \leq \|h(t)\| = \left\| \int_{t_0}^{t} Ah(s)\, ds \right\| \leq \int_{t_0}^{t} |A|\, \|h(s)\|\, ds \quad (*)$$

for $t \geq t_0$

where $|A| := \max_{z \neq 0} \frac{1}{\|z\|} \|Az\|$ (= constant > 0)

Let $G(t) = \int_{t_0}^{t} |A| \, \|h(s)\| \, ds$ then $\frac{dG}{dt} = |A| \, \|h(t)\|$

and (*) becomes $\frac{1}{|A|} \frac{dG(t)}{dt} - G(t) \leq 0$.

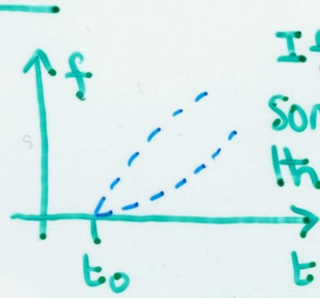
The LHS has integrating factor I.F. $= e^{\int_{t_0}^{t} -|A| ds}$

$= e^{-(t-t_0)|A|}$ since $|A|$ is a constant, and hence can be written as $\frac{d}{dt}\Big(\underbrace{e^{-(t-t_0)|A|} G(t)}_{f(t)} \Big) \leq 0$

Introduce $f(t) := e^{-(t-t_0)|A|} \int_{t_0}^{t} |A| \, \|h(s)\| \, ds$

satisfies $f(t_0) = 0$, $\left.\begin{matrix} f(t) \geq 0 \\ f'(t) \leq 0 \end{matrix}\right\}$ for $t \geq t_0$.

Hence $f(t) \equiv 0$ for $t \geq t_0$.



If $f > 0$ for some $t > t_0$ then $f' > 0$ ⨳.

Hence $G(t) \equiv 0$ for $t \geq t_0$

and (*) gives

$$0 \leq \|h(t)\| \leq 0 \quad \forall \, t \geq t_0 \quad \text{i.e. } \|h(t)\| \equiv 0 \quad \forall \, t \geq t_0.$$

A similar argument holds for $t \leq t_0$ and thus $\|h(t)\| = 0 \; \forall \, t \in \mathbb{R}$ i.e. $h(t) \equiv 0$. □

Definition:

The vector functions $y_i : I \to \mathbb{C}^n$ $i=1,\dots,m$ with $I \subset \mathbb{R}$ some interval are called linearly independent on I if for constants $c_i \in \mathbb{C}$

$$\sum_{i=1}^{m} c_i y_i(t) = 0 \quad \forall t \in I \implies c_1 = c_2 = \dots = c_m = 0$$

The functions y_i are linearly dependent if they are not linearly independent i.e. if $\exists$ constants $c_1, \dots, c_m$ not all zero s.t. $\sum_{i=1}^{m} c_i y_i(t) = 0 \quad \forall t \in I$.

Example: Let $I = [0,1]$, $y_1(t) = \begin{pmatrix} e^t \\ te^t \end{pmatrix}$, $y_2(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$.
Are y_1, y_2 lin. indep. on I?

Solution: Answer is yes. Argue by contradiction:
Assume y_1, y_2 are not lin. indep.
Then $\exists$ c_1, c_2 not both zero s.t.

$$c_1 y_1(t) + c_2 y_2(t) = 0 \quad \forall t \in I$$

i.e. $$\left.\begin{aligned} c_1 e^t + c_2 &= 0 \\ c_1 t e^t + c_2 t &= 0 \end{aligned}\right. \quad \forall t \in I.$$

$$\begin{aligned} \text{Let } t=0: &\quad c_1 + c_2 = 0 \\ \text{Let } t=1: &\quad c_1 e^1 + c_2 = 0 \end{aligned} \Bigg\} \; c_1 = c_2 = 0 \quad ⨳$$

Hence y_1, y_2 are lin. indep. $\square$

Notes:

1. $y_1(t) = e^t y_2(t)$ i.e. $y_1(t)$ is not a constant multiple of $y_2(t)$ $\forall\, t \in I$.
   Thus $y_1(t), y_2(t)$ can't be lin. dep.

2. General Principle: To show linear independence assume linear dependence and seek a contradiction.