1.3. Linearly Independent Solutions

Consider the homogeneous autonomous system

$$\dot{x} = A x \qquad (1.5)$$

where $A \in \mathbb{C}^{n \times n}$ constant matrix and $x = x(t) \in \mathbb{C}^n$

Theorem: There exist n lin. indep. (l.i.) solutions $x_i : \mathbb{R} \to \mathbb{C}^n$ of (1.5). Any other solution $x(t)$ can be written

$$x(t) = \sum_{i=1}^{n} c_i x_i(t) \qquad c_i \in \mathbb{C} \qquad (1.6)$$

Proof: Let $v_1, \dots, v_n \in \mathbb{C}^n$ be n lin. indep. vectors.

Let $x_i(t)$ be unique solutions of the IVPs (see Theorem in §1.2)

$$\dot{x}_i = A x_i \,, \quad x_i(t_0) = v_i \quad i = 1, \dots, n.$$

By Qn 4(a) Problem Sheet 1 the functions $x_i(t)$ are l.i.

Let $x(t)$ be an arbitrary solution of (1.5) with say $x(t_0) = x_0$.

Then $\exists$ constants $c_1, \dots, c_n \in \mathbb{C}$ s.t. $x_0 = \sum_{i=1}^{n} c_i v_i$ (since $\{v_i\}_{i=1}^{n}$ form a basis for $\mathbb{C}^n$).

Define $y(t) := \sum_{i=1}^{n} c_i x_i(t)$

Then $\dot{y} = \sum_{i=1}^{n} c_i \dot{x}_i = \sum_{i=1}^{n} c_i A x_i = A y$

and $y(t_0) = x_0$ i.e. $y(t)$ satisfies same IVP as $x(t)$.

By uniqueness (of IVP) $y(t) = x(t)$.

$\square$

How to find n lin. indep. solutions:

Seek solutions of the form

$$x(t) = e^{\lambda t} v \qquad (1.7)$$

where $v \in \mathbb{C}^n$ constant vector. Then

$$\dot{x}(t) = \lambda e^{\lambda t} v$$

If $x(t)$ defined in (1.7) is to satisfy $\dot{x} = Ax$ then we require

$$\lambda e^{\lambda t} v = A e^{\lambda t} v = e^{\lambda t} A v \iff Av = \lambda v.$$

Hence $x(t)$ given by (1.7) is a solution of (1.5) iff λ is an eigenvalue of A and v a corresponding eigenvector (sometimes called an ordinary eigenvector).

Example: $\dot{x} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$ (1.8)

Eigenvalues of $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = 0 \Rightarrow 1-\lambda = \pm 2$$
$$\Rightarrow \lambda = 3 \text{ or } -1.$$

Corresponding eigenvectors:

For $\lambda = 3$: $Av = 3v \Rightarrow (A - 3I)v = 0$

$$\therefore \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \underbrace{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}_{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{i.e.} \quad \begin{array}{l} -2a_1 + a_2 = 0 \\ 4a_1 - 2a_2 = 0 \end{array}$$

Thus $v = \begin{pmatrix} a_1 \\ 2a_1 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\forall a_1 \in \mathbb{C}$ (wlog take $a_1 = 1$)

Hence $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 3$.

For $\lambda = -1$: $Av = -v \Rightarrow (A + I)v = 0$ i.e. $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore \left.\begin{array}{l} 2b_1 + b_2 = 0 \\ 4b_1 + 2b_2 = 0 \end{array}\right\} \quad v = \begin{pmatrix} b_1 \\ -2b_1 \end{pmatrix} = b_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \forall b_1 \in \mathbb{C}$$

Wlog take $b_1 = 1$, then $v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector for $\lambda = -1$

Thus $x_1(t) = e^{3t}\begin{pmatrix}1\\2\end{pmatrix}$, $x_2(t) = e^{-t}\begin{pmatrix}1\\-2\end{pmatrix}$ are solutions of (1.8).

Further, since v_1, v_2 are l.i. then $x_1(t), x_2(t)$ are l.i.

Thus the general solution of (1.8) is

$$x(t) = c_1 x_1(t) + c_2 x_2(t)$$
$$= c_1 e^{3t}\begin{pmatrix}1\\2\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\-2\end{pmatrix}$$

for arbitrary constants $c_1, c_2 \in \mathbb{C}$.

$\square$

Corollary: Let A be a constant $n \times n$ matrix. If A has n lin. indep. eigenvectors then the general solution of $\dot{x} = Ax$ is

$$x(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} v_i$$

where λ_i are eigenvalues of A and v_i are corresponding lin. indep. eigenvectors.

Notation: For a constant $n \times n$ matrix A, the characteristic polynomial will be denoted by

$$\pi_A(\lambda) = \det(A - \lambda I) .$$

The set of eigenvalues (spectrum) of A is

$$\operatorname{spec}(A) = \{ \lambda \in \mathbb{C} : \pi_A(\lambda) = 0 \} .$$

Some Linear Algebra Facts:

A has n distint eigenvalues

$\Downarrow$

A symmetric $n \times n$ $\Rightarrow$ A has n lin. indep. eigenvectors

$\Updownarrow$

A is diagonalisable

- Eigenvectors corresponding to distinct eigenvalues of A are lin. indep.