Corollary: Let $\Phi(t)$ be a fundamental matrix of $\dot{x} = Ax$. Then

$$\exp(tA) = \Phi(t)\,\Phi(0)^{-1}.$$

Proof: By the Lemma in §1.4, $\exists$ a constant matrix $C \in \mathbb{C}^{n\times n}$ s.t.

$$\exp(tA) = \Phi(t)\,C$$

since $\exp(tA)$ is a fundamental matrix.

For $t=0$, $\exp(0A) = I \Rightarrow I = \Phi(0)\,C$

$$\therefore\ C = \Phi(0)^{-1}. \qquad \square$$

Example: For $\dot{\Phi} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \Phi$, we know from the example in §1.4 that

$$\Phi(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$ is a fundamental matrix.

Moreover

$$\Phi(t)\,\Phi(0)^{-1} = \begin{pmatrix} \frac{1}{2}e^{3t}+\frac{1}{2}e^{-t} & \frac{1}{4}e^{3t}-\frac{1}{4}e^{-t} \\ e^{3t}-e^{-t} & \frac{1}{2}e^{3t}+\frac{1}{2}e^{-t} \end{pmatrix}$$ from the previous example

$$= \exp\left(t\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\right) = \exp\left(\begin{pmatrix} t & t \\ 4t & t \end{pmatrix}\right)$$

$\square$

Key Result: If we know a fundamental matrix then $\exp(tA)$ can be computed.

Alternative View: Computing the matrix exponential via diagonalisation.

If $A \in \mathbb{C}^{n\times n}$ is diagonalisable, then $\exists$ an invertible $T \in \mathbb{C}^{n\times n}$ s.t.

$$T^{-1}AT = D \equiv \operatorname{diag}(\lambda_1, \dots, \lambda_n) \equiv \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Thus $A = TDT^{-1}$ and

$$\exp(tA) = \exp(tTDT^{-1}) = \sum_{k=0}^{\infty} \frac{t^k}{k!}(TDT^{-1})^k$$

$$\overset{(\dagger)}{=} \sum_{k=0}^{\infty} \frac{t^k}{k!} TD^kT^{-1} = T\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} D^k\right)T^{-1}$$

$$= T\exp(tD)T^{-1}$$

$$\overset{(\dagger\dagger)}{=} T\begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix}T^{-1}$$

$\square$

⌈ Notes: T is composed of the n lin. indep. eigenvectors of A.

$(\dagger)$: $(TDT^{-1})^k = (TDT^{-1})(TDT^{-1})\cdots(TDT^{-1})$

$$= T\underbrace{DD\cdots D}_{k \text{ terms}}T^{-1} = TD^kT^{-1}$$

$(\dagger\dagger)$: $\exp(tD) = \exp(t\,\mathrm{diag}(\lambda_1, \dots, \lambda_n))$

$$= \mathrm{diag}(e^{t\lambda_1}, \dots, e^{t\lambda_n})$$

$$= \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix}$$ (See Sheet 4 Qn 3).

⌋

Example: For $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ we calculated $\lambda_1 = 3$ with $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\lambda_2 = -1$ with $v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ as the eigenvalues and corresponding eigenvectors. Write $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$ as one matrix equation:

$$A \underbrace{\left(\begin{array}{c|c} \vdots & \vdots \\ v_1 & v_2 \\ \vdots & \vdots \end{array} \right)}_{T} = \underbrace{\left(\begin{array}{c|c} \vdots & \vdots \\ v_1 & v_2 \\ \vdots & \vdots \end{array} \right)}_{T} \underbrace{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}}_{D} \quad \text{i.e. } A = TDT^{-1}$$

Thus
$$\exp(tA) = -\frac{1}{4} \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{pmatrix}$$

$\square$

Note: If A is not diagonalisable, it can still be represented in the form $A = TJT^{-1}$ where J is Jordan normal form (a nearly diagonal matrix). (See further comments in typed Course Notes.)

The following Theorem (from Linear algebra) ensures $\exists\ n$ lin. indep. (generalised) eigenvectors.

Theorem: (Primary Decomposition Theorem)
Let $A \in \mathbb{C}^{n \times n}$ and let $\pi_A(\lambda) = \prod_{j=1}^{\ell} (\lambda_j - \lambda)^{m_j}$ with λ_j distinct eigenvalues of A and $\sum_{j=1}^{\ell} m_j = n$.
Then, for each $j = 1, \ldots, \ell$, $\exists\ m_j$ lin. indep. (generalised) eigenvectors wrt λ_j. The combined set of n generalised eigenvectors is linearly independent.

1.6 Non-homogeneous Systems.

Let $A \in \mathbb{C}^{n \times n}$ and consider the non-homogeneous system

$$\dot{x}(t) = A\,x(t) + g(t) \qquad (1.11)$$

for $x : \mathbb{R} \to \mathbb{C}^n$ and $g : \mathbb{R} \to \mathbb{C}^n$. Let $\Phi(t)$ be a fundamental matrix of the corresponding homogeneous system $\dot{x} = Ax$.

Seek a solution to (1.11) in the variation of parameters form $x(t) = \Phi(t)\,u(t)$ for some function $u : \mathbb{R} \to \mathbb{C}^n$.

Then $\dot{x} = \dot{\Phi}u + \Phi\dot{u} = A\Phi u + \Phi\dot{u} = Ax + \Phi\dot{u}$

But $\dot{x} = Ax + g \implies \dot{u} = \Phi^{-1}g$

$$\therefore\ u(t) = \int_{t_0}^{t} \Phi^{-1}(s)\,g(s)\,ds + C \qquad C \in \mathbb{C}^n \text{ is a constant matrix.}$$

$$\therefore\ x(t) = \Phi(t)\,C + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\,g(s)\,ds \qquad (1.12)$$

is the general solution.

Given the IVP

$$\dot{x} = Ax + g, \quad x(t_0) = x_0 \in \mathbb{C}^n \qquad (1.13)$$

C can be determined in (1.12):

$$x(t_0) = \Phi(t_0)\, C + \underbrace{\Phi(t_0) \int_{t_0}^{t_0} \Phi^{-1}(s)\, g(s)\, ds}_{=0}$$

$$\therefore \quad C = \Phi^{-1}(t_0)\, x_0 \,.$$

Thus the solution of the IVP (1.13) is

$$\boxed{x(t) = \Phi(t)\,\Phi^{-1}(t_0)\, x_0 + \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)\, g(s)\, ds \qquad (1.14)}$$

The solution (1.14) (and (1.12)) are said to have been obtained by variation of parameters.

[Exercise: Check directly (by differentiation) that (1.14) and (1.12) satisfy (1.11).]