Example: Solve the IVP

$$\dot{x}_1 = -x_1 - x_2 + e^t \quad , \quad x_1(0) = 0$$
$$\dot{x}_2 = -3x_1 + x_2 \quad , \quad x_2(0) = 2.$$

Solution: In matrix form:

$$\underbrace{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}}_{\dot{x}} = \underbrace{\begin{pmatrix} -1 & -1 \\ -3 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} e^t \\ 0 \end{pmatrix}}_{g} , \quad \underbrace{\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}}_{x(0)} = \underbrace{\begin{pmatrix} 0 \\ 2 \end{pmatrix}}_{x_0}$$

Now $\pi_A(\lambda) = (-1-\lambda)(1-\lambda) - 3 = \lambda^2 - 4$

and thus $\mathrm{spec}(A) = \{-2, 2\}$.

For $\lambda = -2$: $(A - \lambda I)v_1 = 0 \overset{(*)}{\Longrightarrow} v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For $\lambda = 2$: $(A - \lambda I)v_2 = 0 \overset{(*)}{\Longrightarrow} v_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ ((*) exercise)

Hence $\Phi(t) = \begin{pmatrix} e^{2t} & e^{-2t} \\ -3e^{2t} & e^{-2t} \end{pmatrix}$ is a fundamental matrix of $\dot{x} = Ax$,

and $\Phi^{-1}(t) = \frac{1}{4}\begin{pmatrix} e^{-2t} & -e^{-2t} \\ 3e^{2t} & e^{2t} \end{pmatrix}$.

Hence $\Phi^{-1}(0)\,x_0 = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}$

and thus the (variation of parameters) solution (1.14) gives

$$x(t) = \begin{pmatrix} e^{2t} & e^{-2t} \\ -3e^{2t} & e^{-2t} \end{pmatrix}\begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} e^{2t} & e^{-2t} \\ -3e^{2t} & e^{-2t} \end{pmatrix}\int_0^t \frac{1}{4}\begin{pmatrix} e^{-2s} & -e^{-2s} \\ 3e^{2s} & e^{2s} \end{pmatrix}\begin{pmatrix} e^{s} \\ 0 \end{pmatrix} ds$$

$$= \frac{1}{2}\begin{pmatrix} -e^{2t}+e^{-2t} \\ 3e^{2t}+e^{-2t} \end{pmatrix} + \frac{1}{4}\begin{pmatrix} e^{2t} & e^{-2t} \\ -3e^{2t} & e^{-2t} \end{pmatrix}\int_0^t \begin{pmatrix} e^{-s} \\ 3e^{3s} \end{pmatrix} ds$$

$$= \frac{1}{2}\begin{pmatrix} -e^{2t}+e^{-2t} \\ 3e^{2t}+e^{-2t} \end{pmatrix} + \frac{1}{4}\begin{pmatrix} e^{2t} & e^{-2t} \\ -3e^{2t} & e^{-2t} \end{pmatrix}\begin{pmatrix} -e^{-t}+1 \\ e^{3t}-1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -e^{2t}+e^{-2t} \\ 3e^{2t}+e^{-2t} \end{pmatrix} + \frac{1}{4}\begin{pmatrix} -e^{t}+e^{2t}+e^{t}-e^{-2t} \\ 3e^{t}-3e^{2t}+e^{t}-e^{-2t} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -e^{2t}+e^{-2t} \\ 3e^{2t}+e^{-2t} \end{pmatrix} + \frac{1}{4}\begin{pmatrix} e^{2t}-e^{-2t} \\ 4e^{t}-3e^{2t}-e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{4}e^{2t}+\frac{1}{4}e^{-2t} \\ \frac{3}{4}e^{2t}+e^{t}+\frac{1}{4}e^{-2t} \end{pmatrix}.$$

$\square$

⌈Exercise: Check directly from this last expression that $x(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $\dot{x} = Ax + g$ are satisfied.⌋

2. Laplace Transform.

2.1 Definition and Basic Properties.

Definition: The Laplace transform (LT) of a function $f:[0,\infty)\to\mathbb{R}$ is the complex-valued function of the complex variable s defined by

$$\mathcal{L}\{f(t)\}(s) = \hat{f}(s) := \int_0^\infty f(t)\, e^{-st}\, dt \qquad (2.1)$$

Remarks:

1. If $g:\mathbb{R}\to\mathbb{C}$ is a complex valued function, then

$$\int g(t)\,dt = \int \mathrm{Re}\, g(t)\,dt + i\int \mathrm{Im}\, g(t)\,dt.$$

2. For $z\in\mathbb{C}$ we have

$$|e^z| = |e^{\mathrm{Re}\, z + i\,\mathrm{Im}\, z}| = |e^{\mathrm{Re}\, z}|\,|e^{i\,\mathrm{Im}\, z}| = e^{\mathrm{Re}\, z}$$

3. Not all functions possess Laplace transforms. For the *improper* integral (2.1) to make sense, we need:

Definition: A function $f : [0,\infty) \to \mathbb{R}$ is of *exponential order* if $\exists$ constants $\alpha, M \in \mathbb{R}$ with $M > 0$ s.t.

$$|f(t)| \leq M e^{\alpha t} \qquad \forall\, t \in [0,\infty). \qquad \square$$

Note that the LT of a function may only be defined for certain values of the variable s.

Theorem 2.1: Suppose $f : [0,\infty) \to \mathbb{R}$ is piecewise continuous and of exponential order with constants α, M. Then $\mathcal{L}\{f\}(s)$ exists $\forall\, s \in \mathbb{C}$ with $\operatorname{Re} s > \alpha$.

Proof: See Typed Course Notes - not examinable.

Example: Consider $f(t) = e^{ct}$, $c \in \mathbb{C}$

Then $f(t)$ is of exponential order:

$$|f(t)| = |e^{ct}| = e^{Re(c)t} \leq M e^{\alpha t} \quad \text{with } M=1, \; \alpha = Re(c).$$

Does the improper integral of its LT exist?

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{ct} e^{-st}\, dt = \lim_{T\to\infty} \int_0^T e^{(c-s)t}\, dt$$

$$= \lim_{T\to\infty} \left[\frac{e^{(c-s)t}}{(c-s)} \right]_{t=0}^{t=T}$$

$$= \lim_{T\to\infty} \frac{1}{(c-s)} \left(e^{(c-s)T} - 1 \right)$$

If $Re(s) > Re(c)$ then $\lim_{T\to\infty} e^{(c-s)T} = 0$

since $e^{(c-s)T} = e^{(Re(c)-Re(s))T} e^{i(Im(c)-Im(s))T}$

Thus $\mathcal{L}\{e^{ct}\}(s) = \frac{1}{s-c}$ for $Re(s) > Re(c)$ (2.2)

In particular for $c=0$:

$$\mathcal{L}\{1\}(s) = \frac{1}{s} \quad \text{for } Re(s) > 0.$$

□

Theorem 2.2: Suppose $f(t)$ and $g(t)$ are of exponential order. Then for $\mathrm{Re}(s)$ sufficiently large, the following properties hold:

1. Linearity: For $a, b \in \mathbb{C}$

$$\mathcal{L}\{a f(t) + b g(t)\}(s) = a\hat{f}(s) + b\hat{g}(s).$$

2. Transform of a derivative: Let $f' = \frac{df}{dt}$

$$\mathcal{L}\{f'(t)\}(s) = s\hat{f}(s) - f(0) \qquad (2.3)$$

3. Transform of an integral:

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}(s) = \frac{1}{s}\hat{f}(s) \qquad (2.4)$$

4. Damping formula:

$$\mathcal{L}\{e^{-at} f(t)\}(s) = \hat{f}(s+a)$$

5. Delay formula: For $T > 0$

$$\mathcal{L}\{f(t-T)\,H(t-T)\}(s) = e^{-sT}\hat{f}(s)$$

where $H(t) = \begin{cases} 0, & t \le 0 \\ 1, & t > 0 \end{cases}$ Heaviside step function.

Remarks:

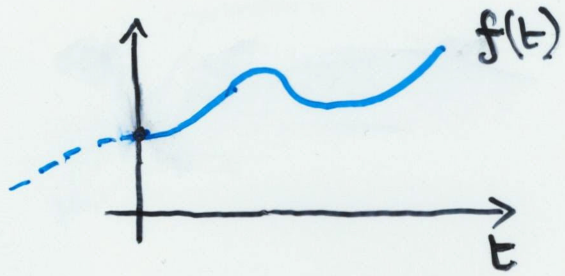
- Replacing f by f' in (2.3) gives

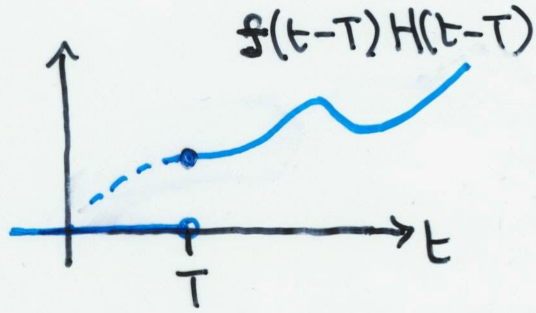
$$\mathcal{L}\{f''(t)\}(s) = s\mathcal{L}\{f'(t)\}(s) - f'(0)$$
$$= s^2\hat{f}(s) - sf(0) - f'(0)$$

and more generally for the nth derivative:

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n\hat{f}(s) - \left[s^{n-1}f(0) + s^{n-2}f'(0) + \cdots + f^{(n-1)}(0)\right].$$

- In 5, if $f(t)$ is undefined for $t<0$, then set $f(t)=0$ for $t<0$.





i.e. start is delayed. coincides with $f(t-T)$ for $t \geq T$ and vanishes for $0 \leq t < T$.