Revision

Cover:

1. Notes from Lectures
2. Problem Sheets (all 11)
3. Past Exam Papers (last 5 years)

Exam:

- Same format as previous years (4 Qns)
- 2 Qns - Systems of Linear ODEs (Part 1)

  1 Qn - LT / Control (Parts 2 & 3)

  1 Qn - Control (Part 3).

  (Approximate break down).

MA20220 - Revision Notes (Parts 1 and 2).

Part 1: Systems of linear ODEs.

1.1 First-order Systems.

- Any linear ODE (of any order) can be written as a first-order system
- Standard form: $\dot{x}(t) = A(t)\,x(t) + g(t)$.
  $A \in \mathbb{C}^{n\times n}$, $x, g \in \mathbb{C}^n$, $t \in \mathbb{R}$.
- Only consider A constant matrix
  i.e. homogeneous system $\dot{x}(t) = A\,x(t)$
  is autonomous.

1.2 Autonomous Homogeneous Systems

- **Theorem** (Existence & Uniqueness of IVPs):
  Let $A \in \mathbb{C}^{n\times n}$, $x_0 \in \mathbb{C}^n$. Then the IVP

$$\dot{x}(t) = A\,x(t), \quad x(t_0) = x_0 \qquad (1.4)$$

has a unique solution.

- **Proof** (Main ideas):

  - Existence: $x(t) = \exp((t-t_0)A)\, x_0$ satisfies (1.4).

  matrix exponential: For $Y \in \mathbb{C}^{n\times n}$

$$\exp(tY) = \sum_{k=0}^{\infty} \frac{1}{k!}(tY)^k \equiv I + tY + \frac{t^2}{2}Y^2 + \dots$$

  - Uniqueness: Let $y(t)$ also satisfy (1.4).
  Consider $h(t) = x(t) - y(t)$ then

$$\dot{h}(t) = A\,h(t), \quad h(t_0) = 0$$

  We showed the only solution has $h(t) = 0 \ \forall t$.

$\square$

- **Definition**

- The vector functions $\{y_i(t)\}_{i=1}^{m}$ $t \in I \subset \mathbb{R}$ are linearly independent on I if for constants $c_i \in \mathbb{C}$, $\sum_{i=1}^{m} c_i y_i(t) = 0 \ \forall t \in I \Rightarrow c_1 = c_2 = \dots = c_m = 0$.

- The functions $\{y_i(t)\}_{i=1}^{m}$ are linearly dependent if they are not linearly independent i.e. if $\exists$ constants $c_1, \dots, c_m$ not all zero s.t. $\sum_{i=1}^{m} c_i y_i(t) = 0$ $\forall t \in I$.

- To show linear independence:
  Assume linear dependence and seek a contradiction.

- Sheet 1 4(a):
  If $\{x_i(t_0)\}_{i=1}^{n}$ are lin. indep. vectors

  then $\{x_i(t)\}_{i=1}^{n}$ are lin. indep. vector functions.

1.3. Linearly Independent Solutions.

- Homogeneous autonomous system: $\dot{x}(t) = A\,x(t)$ (1.5)
  $(A \in \mathbb{C}^{n\times n})$.

- Theorem (Existence of n and only n lin. indep. sol$^{\underline{n}}$s):
  There exist n lin. indep. solutions $\{x_i(t)\}_{i=1}^{n}$

  of (1.5). Any other solution $x(t)$ can be written

$$x(t) = \sum_{i=1}^{n} c_i\,x_i(t) \qquad (1.6)$$

for constants $c_i \in \mathbb{C}$. $\square$

- **Corollary:** If A has n lin. indep. eigenvectors then the general solution of (1.5) is

$$x(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} v_i$$

where $v_i \in \mathbb{C}^n$ are lin. indep. eigenvectors with corresponding eigenvalues $\lambda_i \in \mathbb{C}$ and $c_i \in \mathbb{C}$ are arbitrary constants.

$\square$

Note: $(A - \lambda_i I) v_i = 0$.

- **Definitions:**

- Characteristic polynomial of A: $\pi_A(\lambda) = \det(A - \lambda I)$
- Set of eigenvalues (spectrum) of A: $\mathrm{spec}(A) = \{\lambda \in \mathbb{C} \mid \pi_A(\lambda) = 0\}$

- **Linear Algebra Facts:**

$$\begin{array}{ccc} & & A \text{ has } n \text{ distinct eigenvalues} \\ & & \Downarrow \\ A \text{ symmetric } n\times n & \Longrightarrow & \boxed{A \text{ has } n \text{ lin. indep. eigenvectors}} \\ & & \Updownarrow \\ & & A \text{ is diagonalisable.} \end{array}$$

- Eigenvectors corresponding to distinct eigenvalues of A are lin. indep.

- **Definition:** (Generalised Eigenvectors)

For $A \in \mathbb{C}^{n \times n}$ and $\lambda \in \mathrm{spec}(A)$, a vector $v \in \mathbb{C}^n$ is called a generalised eigenvector wrt λ of order $m \in \mathbb{N}$ if the following hold:

(i) $(A - \lambda I)^k v \neq 0 \quad , \quad 0 \leq k \leq m-1$

(ii) $(A - \lambda I)^m v = 0$.

□

- **Lemma:** (Constructing Generalised Eigenvectors)

Let $\lambda \in \mathrm{spec}(A)$ and v be a generalised eigenvector of order $m \geq 2$. Then for $k = 1, \dots, m-1$ the vector

$$v_{m-k} = (A - \lambda I)^k v$$

is a generalised eigenvector of order $m-k$.

□

- **Theorem:** Let $A \in \mathbb{C}^{n \times n}$ and $v \in \mathbb{C}^n$ be a generalised eigenvector of order m wrt eigenvalue $\lambda \in \mathrm{spec}(A)$. Define m vectors as follows:

$$v_k = (A - \lambda I)^{m-k} v \quad k = 1, \dots, m \quad (v_m = v) .$$

(i) Then $v_1, \dots, v_m$ are lin. indep.

(ii) The functions $x_k(t) = e^{\lambda t} \sum_{i=0}^{k-1} \frac{t^i}{i!} v_{k-i}$

for $k = 1, \dots, m$ form a set of lin. indep. solutions for $\dot{x} = Ax$.

□

- **Definition:** For $A \in \mathbb{C}^{n \times n}$, an eigenvalue $\lambda \in \mathrm{spec}(A)$ has geometric multiplicity $m \in \mathbb{N}$ if m is the largest number for which m lin. indep. eigenvectors exist. (If $m = 1$, λ is said to be a simple eigenvalue.)

  The algebraic multiplicity is the multiplicity of λ as a root of $\pi_A(\lambda)$.

- Generalised eigenvectors are needed if the geometric multiplicity (for a given $\lambda \in \mathrm{spec}(A)$) is less than the algebraic multiplicity.

- For a given $\lambda \in \mathrm{spec}(A)$, the total number of lin. indep. eigenvectors to be found = algebraic multiplicity.

  (generalised if necessary)

1.4. Fundamental Matrices

- Fundamental System: n lin. indep. solutions $x_1(t), \dots, x_n(t)$ of $\dot{x} = Ax$ $A \in \mathbb{C}^{n \times n}$.

(i) Any other solution $x(t) = \sum_{i=1}^{n} c_i x_i(t)$ $c_i \in \mathbb{C}$ constants

(ii) Fundamental Matrix: $\Phi(t) = (x_1(t) | \cdots | x_n(t))$

- satisfies $\dot{\Phi}(t) = A\Phi(t)$
- $\Phi^{-1}(t)$ exists (since columns lin. indep.).

- Lemma If $\Phi(t)$ and $\Psi(t)$ are fundamental matrices of $\dot{x} = Ax$, then $\exists$ constant matrix $C \in \mathbb{C}^{n \times n}$ s.t. $\Phi(t) = \Psi(t)\, C$ $\forall t \in \mathbb{R}$.

- Theorem $\Phi(t) = \exp(tA)$ is a fundamental matrix for the system $\dot{x} = Ax$

1.5. IVPs Revisited.

- Let $\Psi(t)$ be a fundamental matrix with $\Psi(t_0) = \Psi_0$.

- <u>Claim</u>: $\Phi(t) = \Psi(t)\,\Psi_0^{-1}\,\Phi_0$ solves the IVP

$$\dot{\Phi} = A\,\Phi\,, \quad \Phi(t_0) = \Phi_0 \qquad (1.9)$$

$(A \in \mathbb{C}^{n \times n})$. □

- <u>Corollary</u>: The unique solution of the IVP (1.9) is

$$\Phi(t) = \exp\big((t-t_0)A\big)\,\Phi_0 \,.$$ □

- <u>Corollary</u>: If $\Phi(t)$ is a fundamental matrix of $\dot{x} = Ax$ then

$$\exp(tA) = \Phi(t)\,\Phi(0)^{-1}\,.$$ □

- (Alternative view: calculate matrix exponential via diagonalisation).

- <u>Theorem</u>: (Primary Decomposition Theorem)

Let $A \in \mathbb{C}^{n \times n}$ and $\pi_A(\lambda) = \prod_{j=1}^{l} (\lambda_j - \lambda)^{m_j}$ with λ_j distinct eigenvalues of A and $\sum_{j=1}^{l} m_j = n$.

Then for each $j = 1, \dots, l$, $\exists$ m_j lin. indep. (generalised) eigenvectors wrt λ_j. The combined set of n generalised eigenvectors is lin. indep. □

1.6. Non-Homogeneous Systems

- Non-homogeneous system: $\dot{x}(t) = A\,x(t) + g(t)$ (1.11)

$A \in \mathbb{C}^{n\times n}$, $x \in \mathbb{C}^n$, $g \in \mathbb{C}^n$.

- Let $\Phi(t)$ be a fundamental matrix of the corresponding (autonomous) homogeneous system $\dot{x} = A\,x$.

- General solution:

$$x(t) = \Phi(t)\,C + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\,g(s)\,ds \qquad (1.12)$$

($C \in \mathbb{C}^n$, arbitrary constant vector)

- IVP: $\dot{x} = Ax + g$, $x(t_0) = x_0 \in \mathbb{C}^n$ (1.13)

has solution

$$\boxed{x(t) = \Phi(t)\,\Phi^{-1}(t_0)\,x_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\,g(s)\,ds} \qquad (1.14)$$

(1.12), (1.14) obtained by "variation of parameters".

Part 2: Laplace Transform.

2.1. Definition & Basic Properties.

• Definition: The Laplace transform (LT) of a function $f:[0,\infty)\to\mathbb{R}$ is the complex-valued function of the complex variable s defined by

$$\mathcal{L}\{f(t)\}(s) = \hat{f}(s) = \int_{t=0}^{\infty} f(t)\,e^{-st}\,dt \qquad (2.1)$$

□

• Definition: A function $f:[0,\infty)\to\mathbb{R}$ is of exponential order if $\exists$ constants $\alpha, M>0 \in \mathbb{R}$ s.t

$$|f(t)| \leq M e^{\alpha t} \quad \forall\, t\in[0,\infty).$$

□

• Theorem: Suppose $f:[0,\infty)\to\mathbb{R}$ is piecewise continuous and of exponential order with constants α, M. Then $\mathcal{L}\{f(t)\}(s)$ exists $\forall\, s\in\mathbb{C}$ with $\mathrm{Re}(s)>\alpha$. □

(Improper integral in (2.1) makes sense).

- **Properties:** For $f(t), g(t)$ exponential order and $\mathrm{Re}(s)$ suff. large:

1. Linearity: $\mathcal{L}\{af(t)+bg(t)\}(s) = a\hat{f}(s)+b\hat{g}(s)$

   $a, b \in \mathbb{C}$ constants

2. Transform of a derivative: $\mathcal{L}\{f'(t)\}(s) = s\hat{f}(s) - f(0)$

   $$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n\hat{f}(s) - \left[s^{n-1}f(0)+s^{n-2}f'(0)+\cdots+f^{(n-1)}(0)\right]$$

3. Transform of an integral: $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}(s) = \frac{1}{s}\hat{f}(s)$

4. Damping formula: $\mathcal{L}\{e^{-at}f(t)\}(s) = \hat{f}(s+a)$

5. Delay formula: $\mathcal{L}\{f(t-T)H(t-T)\}(s) = e^{-sT}\hat{f}(s)$

   $T>0$, $H(t) = \begin{cases} 0 & t\leq 0 \\ 1 & t>0 \end{cases}$ Heaviside step function.

(- **Examples**)

- **Notation**:

  If $\mathcal{L}\{f(t)\}(s) = \hat{f}(s)$ then $\mathcal{L}^{-1}\{\hat{f}(s)\}(t) = f(t)$

  is the inverse Laplace transform.

- Invert using standard transforms.

2.2. Solving ODEs with LT.

- ODE + IC for $x(t)$ $\xrightarrow{\text{Take L.T.}}$ Algebraic Equation for $\hat{x}(s)$

  Algebraic Equation for $\hat{x}(s)$ $\xrightarrow{\text{Inverse L.T. (by standard transforms)}}$ Solution $x(t)$ of ODE + IC

  ODE + IC for $x(t)$ $\dashrightarrow$ Solution $x(t)$ of ODE + IC

- Scalar & Systems of ODEs.

2.3. Convolution Integral.

- Definition: $$(f*g)(t) = \int_{\tau=0}^{t} f(t-\tau)\, g(\tau)\, d\tau$$

is the convolution of f and g. $\big((f*g)(t) = (g*f)(t)\big)$

- $\mathcal{L}\{(f*g)(t)\}(s) = \hat{f}(s)\,\hat{g}(s)$ or $\mathcal{L}^{-1}\{\hat{f}(s)\,\hat{g}(s)\}(t) = (f*g)(t)$.

- LT to calculate matrix exponentials $\exp(tA)$

  - IVP: $\dot{\Phi}(t) = A\,\Phi(t)$, $\Phi(0) = I$ $\qquad A \in \mathbb{C}^{n\times n}$ const. matrix

  - Solution: $\Phi(t) = \exp(tA) = \mathcal{L}^{-1}\{(sI-A)^{-1}\}(t)$

  provided $\mathrm{Re}(s) > \mathrm{Re}(\lambda)$ for each $\lambda \in \mathrm{spec}(A)$.

- **Derivation of variation of parameters formula by LT:**

  - IVP: $\dot{x}(t) = A\,x(t) + g(t)\,,\ x(0) = x_0$

  - Solution:

$$x(t) = \mathcal{L}^{-1}\{(sI-A)^{-1}\}(t)\,x_0 + \mathcal{L}^{-1}\{(sI-A)^{-1}\hat{g}(s)\}(t)$$

$$= \exp(tA)\,x_0 + \exp(tA)\int_{\tau=0}^{t} \exp(-\tau A)\,g(\tau)\,d\tau$$

2.4. Dirac Delta Function

- Def$^{\underline{n}}$: The Dirac Delta "function" $\delta(t)$ is characterised by the following properties:

  (i) $\delta(t) = 0 \quad \forall\ t \in \mathbb{R}\setminus\{0\}$

  (ii) For any function $f: \mathbb{R} \to \mathbb{R}$ that is continuous on an open interval containing 0,

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)\,.$$

- Properties:

1. $\displaystyle\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

2. $\displaystyle\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = f(a)$

3. $\mathcal{L}\{\delta(t)\}(s) = 1$ or $\mathcal{L}^{-1}\{1\}(t) = \delta(t)$

4. $f(t) * \delta(t) = \delta(t) * f(t) = f(t)$

5. $\mathcal{L}\{\delta(t-T)\}(s) = e^{-sT}$ or $\mathcal{L}^{-1}\{e^{-sT}\}(t) = \delta(t-T)$

for $T > 0$

6. $\delta(t-T) * f(t) = f(t-T)\,H(t-T).$

2.5. Final Value Theorem

Theorem: Let $g : [0,\infty) \to \mathbb{R}$ satisfy $|g(t)| \leq M e^{-\alpha t}$ for some $\alpha > 0$, $M > 0$ (i.e. exponentially decaying).

Then

$$\int_{t=0}^{\infty} g(t)\,dt = \lim_{t \to +\infty} (g * H)(t) = \underbrace{\mathcal{L}\{g(t)\}(0)}_{= \hat{g}(0)}$$