On Cavitation, Configurational Forces and Implications for Fracture in a Nonlinearly Elastic Material

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Abstract. Experiments on polymers indicate that large tensile stress can induce cavitation, that is, the appearance of voids that were not previously evident in the material. This phenomenon can be viewed as either the growth of pre-existing infinitesimal holes in the material or, alternatively, as the spontaneous creation of new holes in an initially perfect body. In this paper our approach is to adopt both views concurrently within the framework of the variational theory of nonlinear elasticity. We model an elastomer on a macroscale as a void-free material and, on a microscale, as a material containing certain defects that are the only points at which hole formation can occur. Mathematically, this is accomplished by the use of deformations whose point singularities are constrained. One consequence of this viewpoint is that cavitation may then take place at a point that is not energetically optimal. We show that this disparity will generate configurational forces, a type of force identified previously in dislocations in crystals, in phase transitions in solids, in solidification, and in fracture mechanics.

As an application of this approach we study the energetically optimal point for a solitary hole to form in a homogeneous and isotropic elastic ball subject to radial boundary displacements. We show, in particular, that the center of the ball is the unique optimal point. Finally, we speculate that the configurational force generated by cavitation at a non-optimal material point may be sufficient to result in the onset of fracture. The analysis utilizes the energy-momentum tensor, the asymptotics of an equilibrium solution with an isolated singularity, and the linear theory of elasticity at the stressed configuration that the body occupies immediately prior to cavitation.

Keywords: Asymptotics, cavitation, elastic, energy-momentum, equilibrium, fracture, singular.

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Dedicated to John Ball and Mort Gurtin for their insight and inspiration

1. Introduction

Experiments on elastomers have shown that the application of a sufficiently large tensile load can cause the appearance of holes that were not previously evident in the material. Upon further loading these cavities grow in size and, eventually, coalesce to form cracks. Similar void growth occurs in other materials such as ductile metals and glasses.

The nonlinear theory of elasticity was used by Gent & Lindley [18] to explain the emergence of such holes in elastomers. In their view the application of loads produces local triaxial tensions that cause the enlargement of small pre-existing voids.[1] Although this theory has an amazing degree of agreement with experiment, a material that contains a large number of tiny holes is extremely difficult to deal with from both an analytical and a computational perspective.

These difficulties were partially overcome by the variational approach to cavitation adopted by Ball [3], which does not require pre-existing holes. Instead, a *new* hole will be created in the material (which is considered as initially perfect) whenever it is energetically favorable in reducing the total stored energy. However, the creation of new holes requires deformations that are not continuous and so do not possess a classical derivative at the points of discontinuity. The stored energy must therefore be extended so that it is defined on such deformations, deformations that lie in a Sobolev space. The advantage of this variational approach is that discontinuous Sobolev deformations can be used to interpret void formation through the simultaneous use of two scales: a macroscopic scale (corresponding to the smooth part of the deformation) on which an apparently void-free material undergoes a (classical) finite deformation in response to compression and small tensions; and a microscopic scale, where the material undergoes a singular (discontinuous) deformation, which can be interpreted as the rapid growth of a preexisting microvoid in response to sufficiently large tensions.[2]

One difference between the viewpoints outlined above is that new cavities may form anywhere in the material (if it is modelled as initially perfect), whilst pre-existing holes can only grow at precursors (i.e., flaws in the material). One anticipates that both views should yield the same results provided such precursors are ubiquitous. Alternatively, this difference can be addressed by restricting the set of material points at which void formation can take place. In [47] it is assumed that the number of potential cavitation points is finite and prespecified. It is shown that there are solutions of the resulting equilibrium equations that can exhibit cavitation at such points. These solutions are global minimizers of the energy subject to the constraint that cavitation can only occur at these points. Although the minimization problem contains constraints, the analysis in [47] yields no Lagrange multiplier and consequently no additional resultant forces[3] that arise as a consequence of the constraint.

We show in this paper that the conclusion that no additional forces emerge as a result of restricting the cavitation points is not strictly correct. Instead, the correct conclusion is that no *classical* (Newtonian) forces are created by constraining the cavitation points. The constraint generally results in the creation of nonclassical forces, sometimes called configurational or material forces[4] (see, e.g., [13, 15, 23, 31, 37]), a

[1] See [8, 24] for related approaches in the context of elastoplasticity.

[2] See [26, 45, 49].

[3] For example, the constraint of incompressibility gives rise to a hydrostatic pressure.

[4] We are unaware of any other identification of nontrivial configurational forces *within the context of cavitation in nonlinear elasticity.* Most prior work on such problems (see [27] and the references therein)

type of force that has been previously identified within elasticity[5] in dislocations in crystals [12, 25, 40, 41], fracture mechanics [10, 16, 43], and at phase boundaries [2, 14, 35].

In this paper we propose the view that a material contains a fixed collection of potential cavitation points. When the body is loaded there may then be a mismatch between the points at which the material would prefer[6] to cavitate and the points where cavitation is possible. This disparity results in the emergence of a non-classical force, a force that is due to the desired movement of the cavity to a more appropriate place where the creation of a hole could be accomplished with less energy, provided cavitation could occur there (i.e., if there were a precursor at that point). One can also view this force as the force against the constraint. For cavitation problems this force is concentrated at the point singularities of the deformation.

A potential difference between configurational forces in cavitation and some of the other areas in which they have been identified is that we do not generally[7] anticipate a change in the material location of the cavity after it forms. Relative to the reference configuration, dislocations can accumulate, a crack can propagate, a phase boundary can move through the material, but a cavity has a fixed material location once it is has formed.

Our analysis focuses on a model problem in which a single new hole is created in a homogeneous and isotropic material held by tensile loads on the boundary. We show that if this hole is not created at an optimal material point then a nontrivial configurational force will result and, at least for loads slightly greater the cavitation load, this force will increase as the boundary load increases. This leads to an intriguing possibility: in fracture mechanics configurational force is usually thought to be the driving force for crack propagation and a standard criterion for crack propagation is that this force should exceed a certain threshold. It therefore seems reasonable that one might obtain a solution in which cracks[8] originating at cavitation points can be initiated within a nonlinearly elastic model, as has been observed in experiments.[9]

Before we discuss the details of our results, we make a final comment concerning fracture. In order to simplify our analysis we have, for the most part, restricted our

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has restricted attention to radially symmetric solutions where (see Proposition 2.2) no configurational forces are present.

[5] Such forces have also been identified in solidification. See, e.g., [23] and the references therein.

[6] We assume that the material prefers to cavitate at a point whenever such a singular deformation results in a reduction in energy. Such preferred points are possibly the points of maximum stress. In the view of [32, 33] (cf. [28, 29]) if one considers the local constitutive relation, which will vary from point to point in an inhomogeneous material, one can generally use such an energy criterion to obtain, at each material point, a *cavitation surface* in strain space (the symmetric matrices). It is then energetically favorable for a hole to form whenever the strain at that point crosses this surface.

[7] Of course a cavity, as well as a crack, can heal under certain circumstances.

[8] True fracture would necessitate an alternate interpretation for the derivative, which would no longer lie in the indicated Sobolev space.

[9] Recent research on the growth of voids in ductile metals suggests that the formation of cracks from voids in such materials may require additional ingredients (see, e.g., [17]). See also [32] and [38].

attention to a purely elastic model. If precursors are sparse then larger configurational forces will ensue and adequate force for fracture may be available from nonlinear elasticity by itself. However, if precursors are widespread the resulting configurational forces may be insufficient to initiate fracture when elasticity alone is considered. Additional energy might then result in a significant increase in the magnitude of such forces. For example, the addition of an energy that is required to initiate cavitation and which varies among pre-existing microvoids (e.g., inversely proportional to the microvoid's initial size) has essentially the same effect as a critical cavitation load that varies with the material point.[10] In particular, a single large microvoid (with negligible energy inhibiting cavitation) that is located sufficiently far from an optimal cavitation point, might cavitate before other closer (but smaller) microvoids and so produce the force necessary for fracture. This may be especially appropriate since the requisite fracture force in fracture mechanics is usually attributed to the energy of creation of new crack surface (see, e.g., [16, 21, 43]).

The specific problem we consider is the displacement[11] boundary-value problem: Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a regular region, which we assume is occupied by a nonlinearly elastic body in its reference configuration. An admissible **deformation** is a differentiable (or weakly differentiable) map $\mathbf{u} : \overline{\Omega} \to \mathbb{R}^n$ that is one-to-one (a.e.), satisfies $\det \nabla\mathbf{u} > 0$ (a.e.), and

$$\mathbf{u}(\mathbf{x}) = \lambda\mathbf{x} \quad \text{for all } \mathbf{x} \in \partial\Omega, \tag{1.1}$$

where $\lambda > 0$ is given. If the material is homogeneous and hyperelastic and there are no body forces then the total elastic energy stored in a body that undergoes such a deformation is given by

$$E(\mathbf{u}) = \int_\Omega W(\nabla\mathbf{u}(\mathbf{x}))\, d\mathbf{x}, \tag{1.2}$$

where $W : \mathrm{M}_+^{n\times n} \to \mathbb{R}$ is the stored-energy function and $\mathrm{M}_+^{n\times n}$ denotes the real $n \times n$ matrices with positive determinant.

The equilibrium equations of nonlinear elasticity are the Euler-Lagrange equations for E. These can take a number of forms depending on the variations taken in the energy functional E. In this paper we will make particular use of the following two forms:

$$(\mathrm{Div}\, \mathbf{S}(\nabla\mathbf{u}(\mathbf{x})))_i = \frac{\partial}{\partial x^\alpha}\left[\frac{\partial W}{\partial F^i_\alpha}(\nabla\mathbf{u}(\mathbf{x}))\right] = \mathbf{0} \quad \text{for } i = 1, 2, \ldots, n, \tag{1.3}$$

[10] Experiments on elastomers by Gent and his coworkers [11, 19, 20] have noted that the critical cavitation load appears to increase with decreasing size of the inclusion (a glass bead) used in their experiments. In order to explain this they hypothesize that an elastomer has a distribution of pre-existing microvoids of different sizes and, in accordance with the analysis of [51], smaller voids have larger surface energy to overcome in order to grow in size. Thus a small glass bead produces the required cavitation strain over a small region in the material and there is less chance of finding a large precursor, with small surface energy, in this region.

[11] Results of [1] indicate that a displacement boundary condition on a portion of the boundary (possibly at an inclusion) is needed for cavitation to be energetically favorable.

where we use the convention of summation over repeated indices. The tensor $\mathbf{S} = \partial W/\partial \mathbf{F}$ is called the Piola-Kirchhoff stress tensor. The second form is the so called energy-momentum form of the equations

$$\operatorname{Div} \mathbf{M}(\nabla \mathbf{u}) = \mathbf{0}, \tag{1.4}$$

where

$$\mathbf{M}(\nabla \mathbf{u}) = W(\nabla \mathbf{u})\mathbf{I} - \nabla \mathbf{u}^{\mathrm{T}} \frac{\partial W}{\partial \mathbf{F}}(\nabla \mathbf{u}), \tag{1.5}$$

is known as the **energy-momentum tensor** and $\mathbf{I}$ denotes the $n \times n$ identity matrix. In component form the equations (1.4) are given by

$$\frac{\partial}{\partial x^\alpha}\left[W(\nabla \mathbf{u}(\mathbf{x}))\delta_\alpha^\beta - \frac{\partial u^k}{\partial x^\beta}\frac{\partial W}{\partial F_\alpha^k}(\nabla \mathbf{u}(\mathbf{x}))\right] = 0, \quad \beta = 1, 2, \ldots, n.$$

Model stored-energy functions for which our results apply have the form

$$W(\mathbf{F}) = k\|\mathbf{F}\|^p + h(\det \mathbf{F}) \quad \text{for all } \mathbf{F} \in \mathrm{M}_+^{n\times n},$$

where $k > 0$, $p \in [1, n)$; $h : \mathbb{R}^+ \to \mathbb{R}^+$ is C^2, strictly convex, and satisfies $h(\delta) \to \infty$ as $\delta \to 0^+$ and $h(\delta)/\delta \to \infty$ as $\delta \to \infty$; $\det \mathbf{F}$ denotes the determinant of $\mathbf{F}$; and $\|\cdot\|$ denotes the Euclidean norm on $n\times n$ matrices: $\|\mathbf{F}\|^2 = \mathbf{F} : \mathbf{F}$ and $\mathbf{F} : \mathbf{G} := \operatorname{trace}(\mathbf{F}^{\mathrm{T}}\mathbf{G})$. However, *we emphasize that the approach and results presented apply to a much wider class of stored-energy functions.*

Now let $\mathbf{x}_0 \in \Omega$, then it is known that for a wide class of stored-energy functions with slow growth (e.g., as above with $p > n-1$), there is a minimizer denoted $\mathbf{u}(\cdot, \mathbf{x}_0, \lambda)$ of E that creates no new holes in $\overline{\Omega} \setminus \{\mathbf{x}_0\}$ and which may or may not create a new hole at the prescribed point $\mathbf{x}_0$ (see [47]). Results of Eshelby [13, 15] show that the *configurational* or *material force* (see, e.g., [23, 31, 37]) on $\mathbf{x}_0$ is given, in particular, by

$$\mathbf{f}(\mathbf{x}_0, \lambda) := \int_{\partial\Omega} \mathbf{M}(\nabla \mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda))\mathbf{n}(\mathbf{x})\, dS_{\mathbf{x}}, \tag{1.6}$$

where $\mathbf{n}$ is the outward unit normal to the boundary. We first note in Section 2 that $-\mathbf{f}(\mathbf{x}_0, \lambda) \cdot \mathbf{m}$ is also the derivative of the energy with respect to any one-parameter family of inner variations that translates $\mathbf{x}_0$ in the direction $\mathbf{m}$, that is,

$$-\mathbf{f}(\mathbf{x}_0, \lambda) \cdot \mathbf{m} = \delta E := \left.\frac{d}{dt} E(\mathbf{u}_t)\right|_{t=0}, \tag{1.7}$$

where

$$\mathbf{u}_t(\mathbf{x}) = \mathbf{u}_0(\mathbf{h}_t^{-1}(\mathbf{x})), \qquad \mathbf{h}_t(\mathbf{x}) = \mathbf{x} + t\mathbf{v}(\mathbf{x}),$$

and $\mathbf{v} \in C_0^1(\Omega; \mathbb{R}^n)$ satisfies $\mathbf{v}(\mathbf{x}) \equiv \mathbf{m}$ in some neighborhood of $\mathbf{x}_0$.

Suppose now that $B = B_1(\mathbf{0})$, the unit ball centered at the origin and let $\mathbf{x}_0 = \mathbf{0}$. Then results of [3] for the class of radial deformations

$$\mathbf{u}(\mathbf{x}) = r(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|} \quad \text{for } \mathbf{x} \in B, \tag{1.8}$$

where $r : [0,1] \to [0,\infty)$, show that

PROPOSITION 1.1. *For each $\lambda > 0$ there exists a unique minimizer $\mathbf{u}^{(r)}$ of the energy functional E among radial deformations in $W^{1,p}(B;\mathbb{R}^n)$, $1 \le p < n$, that satisfy $\mathbf{u}(\mathbf{x}) = \lambda\mathbf{x}$ for $\mathbf{x} \in \partial B$. Moreover, there is a critical value $\lambda_{\text{crit}} > 0$ with the property that*

(i) If $\lambda \le \lambda_{\text{crit}}$ the unique radial minimizer $\mathbf{u}^{(r)}$ is the homogeneous deformation $\mathbf{u}^h(\mathbf{x}) \equiv \lambda\mathbf{x}$.

(ii) If $\lambda > \lambda_{\text{crit}}$ the unique radial minimizer $\mathbf{u}^{(r)}$ corresponds to a map of the form (1.8) satisfying $r(0) > 0$.

Thus for $\lambda > \lambda_{\text{crit}}$ the deformation $\mathbf{u}^{(r)}$ is discontinuous and produces a hole of radius $r(0)$ at the center of the ball (this is the phenomenon of cavitation).

The basic assumptions we make on the family of minimizers are drawn from [48]. We first assume that for each $\mathbf{x}_0 \in \Omega$ the corresponding minimizer $\mathbf{u}(\cdot, \mathbf{x}_0, \lambda)$ is unique and that in the case $\Omega = B$ and $\mathbf{x}_0 = \mathbf{0}$ the minimizer $\mathbf{u}(\cdot, \mathbf{0}, \lambda)$ is the radial minimizer given above. Then, by a scaling argument (see, e.g., [48, Lemma 1.2]), $\lambda = \lambda_{\text{crit}}$ is the infimum of the values of λ for which the corresponding minimizer is discontinuous for **any** $\mathbf{x}_0 \in \Omega$.

We then write

$$\mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda) = \lambda\mathbf{x} + \mathbf{w}(\mathbf{x}, \mathbf{x}_0, \lambda)$$

and assume that

$$\mathbf{w}(\cdot, \mathbf{x}_0, \lambda) \to 0 \quad \text{as} \quad \lambda \to \lambda_{\text{crit}}$$

in $C^2(\Omega \setminus B_\delta(\mathbf{x}_0))$ for any $\delta > 0$ sufficiently small, where $B_\delta(\mathbf{x}_0)$ denotes the ball of radius δ centered at $\mathbf{x}_0$. We expand the expression (1.6) in a series in $(\lambda - \lambda_{\text{crit}})$ to obtain

$$\mathbf{f}(\mathbf{x}_0, \lambda) = -\tfrac{1}{2}(\lambda - \lambda_{\text{crit}})^2 \int_{\partial\Omega} \mathbf{n}\nabla_{\mathbf{x}}\dot{\mathbf{w}} : \mathbb{C}[\nabla_{\mathbf{x}}\dot{\mathbf{w}}]\, dS_{\mathbf{x}} + o(|\lambda - \lambda_{\text{crit}}|^2),$$

where

$$\dot{\mathbf{w}}(\mathbf{x}, \mathbf{x}_0) = \frac{\partial}{\partial\lambda}\mathbf{w}(\mathbf{x}, \mathbf{x}_0, \lambda)\Big|_{\lambda=\lambda_{\text{crit}}} \qquad \text{and} \qquad \mathbb{C} = \frac{\partial^2 W}{\partial \mathbf{F}^2}(\lambda_{\text{crit}}\mathbf{I})$$

is the elasticity tensor at the (stressed) configuration $\mathbf{u}^h_{\text{crit}}(\mathbf{x}) \equiv \lambda_{\text{crit}}\mathbf{x}$.

If $\Omega = B$ and $\mathbf{x}_0 = \mathbf{0}$ it is easy to show that $\mathbf{f}(\mathbf{0}, \lambda) = \mathbf{0}$. Then, using an ansatz for the minimizers (see (4.1)), we are able to prove that, in three-dimensions, the configurational force on $\mathbf{x}_0$ satisfies

$$\mathbf{f}(\mathbf{x}_0, \lambda) = -\Psi(|\mathbf{x}_0|, \lambda)\mathbf{x}_0, \quad \Psi(t, \lambda) = (\lambda - \lambda_{\text{crit}})^2\psi(t) + o(|\lambda - \lambda_{\text{crit}}|^2),$$

where $\psi > 0$. Consequently, if $\mathbf{x}_0 \neq \mathbf{0}$ then $\mathbf{f}(\mathbf{x}_0, \lambda) \neq \mathbf{0}$ for $\lambda - \lambda_{\text{crit}} > 0$ and sufficiently small (see Theorem 6.1).

Finally, we note that our results have an interpretation that does not involve configurational forces. Suppose one is interested in determining the point(s) in a body where cavitation is *optimal*, i.e., a point $\mathbf{x}_0 \in \Omega$ if one exists that achieves the infimum:

$$\inf_{\mathbf{x}_0 \in \Omega} E(\mathbf{u}(\cdot, \mathbf{x}_0, \lambda))$$

for all $\lambda - \lambda_{\text{crit}} > 0$ and sufficiently small. Then our results imply that (see Corollary 6.2) in the case when $\Omega = B$ every $\mathbf{x}_0 \neq \mathbf{0}$ is **not** optimal due to the fact that cavitation at points closer to the origin is energetically more favorable than cavitation at points farther away. Consequently, the center of the ball appears to be the only optimal cavitation point.

2. Inner Variations; The Force on a Defect

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a regular region occupied by a homogeneous hyperelastic body in a homogeneous reference configuration. Suppose that $\mathcal{D} \subset \Omega$ is a closed region that may contain one or more defects[12]. Suppose further that $\mathbf{u}_0 : \overline{\Omega} \to \mathbb{R}^n$ is a deformation with finite total elastic energy, i.e.,

$$E(\mathbf{u}_0) := \int_\Omega W(\nabla \mathbf{u}_0(\mathbf{x}))\, d\mathbf{x} < \infty,$$

that is at equilibrium in $\Omega \backslash \mathcal{D}$, i.e., $\mathbf{u}$ satisfies (1.3) and consequently (1.4) in $\Omega \backslash \mathcal{D}$. Then results of Eshelby [13, 14, 15] show that the *configurational force*, $\mathbf{f}(\mathcal{D})$, that $\Omega \backslash D$ exerts on $\mathcal{D}$ is given, in particular, by

$$\mathbf{f}(\mathcal{D}) = \int_{\partial\Omega} \mathbf{M}(\nabla \mathbf{u}_0)\mathbf{n}\, dS. \tag{2.1}$$

Remark. More generally, it is well-known that (2.1) is satisfied when $\partial\Omega$ is replaced by any smooth surface that encloses $\mathcal{D}$. Slightly modified versions of (1.4) and (2.1) are valid when the material is not homogeneous. See, e.g., [23] or [37].

Next, let $\mathbf{m} \in \mathbb{R}^n$ be a unit vector and let $\mathbf{v} \in C_0^1(\Omega; \mathbb{R}^n)$ satisfy $\mathbf{v}(\mathbf{x}) \equiv \mathbf{m}$ for all $\mathbf{x}$ in some open set that contains $\mathcal{D}$. Then, for t sufficiently small, $\mathbf{h}_t : \overline{\Omega} \to \mathbb{R}^n$

[12] By a defect we mean an imperfection or weakness in the material and not particularly in the atomic structure.

defined by $\mathbf{h}_t(\mathbf{x}) = \mathbf{x} + t\mathbf{v}(\mathbf{x})$ is a diffeomorphism of Ω that rigidly translates $\mathcal{D}$ in the direction $\mathbf{m}$, where $\mathbf{u}_t$ is the one-parameter family of *inner variations*:

$$\mathbf{u}_t(\mathbf{x}) = \mathbf{u}_0(\mathbf{h}_t^{-1}(\mathbf{x})).$$

LEMMA 2.1. *The configurational force satisfies*

$$-\mathbf{f}(\mathcal{D}) \cdot \mathbf{m} := \delta E = \frac{d}{dt} E(\mathbf{u}_t)\Big|_{t=0}. \tag{2.2}$$

Remark. The inner product of the configurational force with any unit vector is thus minus the rate of change of the energy when the defect undergoes an infinitesimal translation in the direction of that vector. This simple result is thus in the spirit of Knowles & Sternberg's [34] derivation (see also [6] and [9]) of the conservation law (1.4) by inner translations.

Proof of Lemma 2.1. Let U be an open set with smooth boundary that satisfies $\mathcal{D} \subset U \subset \overline{U} \subset \Omega$. Assume that $\mathbf{v}$ satisfies $\mathbf{v} \equiv \mathbf{m}$ on $\overline{U}$ so that $\nabla \mathbf{v} \equiv \mathbf{0}$ and hence $\nabla \mathbf{h}_t \equiv \mathbf{I}$ on $\overline{U}$. Then it is a standard result[13] that the rate of change in energy with respect to a one-parameter family of inner variations satisfies

$$\delta E = \frac{d}{dt} E(\mathbf{u}_t)\Big|_{t=0} = \int_{\Omega \setminus U} \nabla \mathbf{v} : \mathbf{M}(\nabla \mathbf{u}_0)\, d\mathbf{x}. \tag{2.3}$$

Next, $\mathbf{v} = 0$ on $\partial\Omega$ and, since $\mathbf{u}_0$ is a smooth equilibrium solution in $\Omega \setminus \mathcal{D}$, $\operatorname{Div} \mathbf{M} = \mathbf{0}$ in $\Omega \setminus \mathcal{D}$. Therefore by the divergence theorem

$$\begin{aligned}
\int_{\Omega \setminus U} \nabla \mathbf{v} : \mathbf{M}(\nabla \mathbf{u}_0)\, d\mathbf{x} &= \int_{\Omega \setminus U} \operatorname{div}[\mathbf{M}^{\mathrm{T}} \mathbf{v}]\, d\mathbf{x} \\
&= \int_{\partial(\Omega \setminus U)} \mathbf{v} \cdot \mathbf{M}(\nabla \mathbf{u}_0)\mathbf{n}\, dS \\
&= \mathbf{m} \cdot \int_{\partial U} \mathbf{M}(\nabla \mathbf{u}_0)\mathbf{n}\, dS \\
&= -\mathbf{m} \cdot \int_{\partial \Omega} \mathbf{M}(\nabla \mathbf{u}_0)\mathbf{n}\, dS,
\end{aligned}$$

which together with (2.2) and (2.3) yield (2.1). (Here $\mathbf{n}$ is the outward unit normal to $\Omega \setminus U$ and so it points into U.) □

[13] See, e.g., [44, p. 240] or [47, Theorem 5.2]. Such proofs require $\mathbf{u}_0 \in C^1(\overline{\Omega} \setminus D)$ or, more generally, that W satisfies (2.5) and $\mathbf{u}_0 \in W^{1,1}(\Omega)$.

2.1. Hypotheses on W

We assume throughout the remainder of this paper that $W \in C^2(\mathrm{M}_+^{n\times n})$ is isotropic and frame indifferent so that for any $\mathbf{Q} \in \mathrm{SO}(n)$ (the $n\times n$ special orthogonal matrices) we have

$$W(\mathbf{FQ}) = W(\mathbf{QF}) = W(\mathbf{F}) \quad \text{for } \mathbf{F} \in \mathrm{M}_+^{n\times n}.$$

We further assume that there is a $C > 0$ such that

$$\left\| \mathbf{F}^{\mathrm{T}} \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}) \right\| \le C[W(\mathbf{F}) + 1] \quad \text{for } \mathbf{F} \in \mathrm{M}_+^{n\times n}. \tag{2.4}$$

Remark. Hypothesis (2.4) can be used to show (see [4, 5, 7]) that any energy minimizer $\mathbf{u} \in W^{1,1}(\Omega; \mathbb{R}^n)$ is a weak solution of the energy-momentum equations. It follows from [5, Proposition 2.3] that if W is frame indifferent and satisfies (2.4) then[14]

$$\left\| \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}) \mathbf{F}^{\mathrm{T}} \right\| \le C[W(\mathbf{F}) + 1] \quad \text{for } \mathbf{F} \in \mathrm{M}_+^{n\times n}. \tag{2.5}$$

and consequently $\mathbf{u}$ is also a weak solution of the equilibrium equations in the deformed configuration: the spatial divergence of the Cauchy stress is zero.

2.2. No configurational forces in radial cavitation

The following elementary result is well-known to researchers in cavitation, but has not, we believe, appeared previously in the literature.

PROPOSITION 2.2. *The radial minimizers* $\mathbf{u}_0 = \mathbf{u}^{(r)}$ *given in Proposition 1.1 satisfy*

$$\mathbf{f}(\mathbf{0}, \lambda) = \int_{\partial B} \mathbf{M}(\nabla \mathbf{u}_0)\mathbf{n}\, dS = \mathbf{0}$$

Thus there is no configurational force in radial cavitation.

Proof. We first note that it is a consequence of our assumptions of frame indifference and isotropy of the stored-energy function W that

$$W(\mathbf{F}) = \Phi(v_1, v_2, \ldots, v_n) \ \text{ for all } \ \mathbf{F} \in \mathrm{M}_+^{n\times n}, \tag{2.6}$$

where Φ is a symmetric function of its arguments and $v_1, v_2, \ldots, v_n$ denote the singular values of $\mathbf{F}$ (i.e., the eigenvalues of $\sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}}$).

[14] A proof similar to [5, Proposition 2.3] shows conversely that (2.5) implies (2.4) for isotropic materials. Thus for frame indifferent, isotropic, stored energy functions conditions (2.4) and (2.5) are equivalent.

Next suppose that

$$\mathbf{u}_0(\mathbf{x}) = \frac{r(R)}{R}\,\mathbf{x}, \quad R = |\mathbf{x}|, \quad r : [0,1] \to [0,\infty)$$

is the radial minimizer given in Proposition 1.1. Then

$$\nabla\mathbf{u}_0(\mathbf{x}) = r'(R)\,\frac{\mathbf{x}\otimes\mathbf{x}}{R^2} + \frac{r(R)}{R}\left[\mathbf{I} - \frac{\mathbf{x}\otimes\mathbf{x}}{R^2}\right]$$

and consequently (see, e.g., [3]) the singular values, $v_1, v_2, \ldots, v_n$, of $\mathbf{F} = \nabla\mathbf{u}_0$ are given by $v_1 = r'(R)$, $v_2 = \ldots = v_n = \frac{r(R)}{R}$. Therefore, in view of (2.6),

$$\mathbf{S}(\nabla\mathbf{u}_0) := \frac{\partial W}{\partial\mathbf{F}}(\nabla\mathbf{u}_0) = \Phi_{,1}(R)\frac{\mathbf{x}\otimes\mathbf{x}}{R^2} + \Phi_{,2}(R)\left[\mathbf{I} - \frac{\mathbf{x}\otimes\mathbf{x}}{R^2}\right]$$

and

$$(\nabla\mathbf{u}_0)^{\mathrm{T}}\,\mathbf{S}(\nabla\mathbf{u}_0) = r'(R)\Phi_{,1}(R)\frac{\mathbf{x}\otimes\mathbf{x}}{R^2} + \frac{r(R)}{R}\Phi_{,2}(R)\left[\mathbf{I} - \frac{\mathbf{x}\otimes\mathbf{x}}{R^2}\right], \tag{2.7}$$

where $\Phi_{,i}$ denotes differentiation of Φ with respect to its i-th argument and $\Phi(R)$ and $\Phi_{,i}(R)$ denote Φ and $\Phi_{,i}$ evaluated at the arguments $v_1 = r'(R)$, $v_2 = \ldots = v_n = \frac{r(R)}{R}$, respectively.

Consequently, by (2.7),

$$(\nabla\mathbf{u}_0(\mathbf{x}))^{\mathrm{T}}\mathbf{S}(\nabla\mathbf{u}_0)\mathbf{n}(\mathbf{x}) = r'(1)\Phi_{,1}(1)\mathbf{n}(\mathbf{x}) \quad \text{for} \quad \mathbf{x}\in\partial B,$$

where $\mathbf{n}(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|}$ is the outward unit normal to ∂B. Finally, from the above calculations we conclude

$$\begin{aligned}\int_{\partial B}\mathbf{M}(\nabla\mathbf{u}_0(\mathbf{x}))\mathbf{n}(\mathbf{x})\,dS_{\mathbf{x}} &= \int_{\partial B}[\Phi(R) - r'(R)\Phi_{,1}(R)]\mathbf{n}(\mathbf{x})\,dS_{\mathbf{x}} \\ &= [\Phi(1) - r'(1)\Phi_{,1}(1)]\int_{\partial B}\mathbf{n}(\mathbf{x})\,dS_{\mathbf{x}} = \mathbf{0},\end{aligned}$$

as claimed. □

3. Families of minimizers with one point of discontinuity

Given $\mathbf{x}_0 \in \Omega$ and $\lambda > 0$, Theorem 4.1 of [47] yields a minimizer[15] $\mathbf{u}(\cdot,\mathbf{x}_0,\lambda)$ of E that satisfies the boundary condition $\mathbf{u}(\mathbf{x},\mathbf{x}_0,\lambda) = \lambda\mathbf{x}$ for $\mathbf{x}\in\partial\Omega$. These minimizers are contained in the Sobolev space $W^{1,1}(\Omega;\mathbb{R}^n)$, are one-to-one a.e., and satisfy

[15] Minimizers also exist when surface energy is included in the model. (See [39] and [47, Theorem 4.2].)

$\det \nabla_{\mathbf{x}}\mathbf{u} > 0$ a.e. Throughout this section we make the following hypotheses on this set of minimizers.

3.1. Hypotheses on the minimizers

(M1) For each $\mathbf{x}_0 \in \Omega$ and $\lambda > 0$ the minimizer $\mathbf{u}(\cdot, \mathbf{x}_0, \lambda)$ of E is unique and this family of minimizers satisfies:

(a) For each $\mathbf{x}_0 \in \Omega$ and $\lambda > 0$

$$\mathbf{u}(\cdot, \mathbf{x}_0, \lambda) \in C^2(\Omega \backslash \{\mathbf{x}_0\}; \mathbb{R}^n) \cap C^1(\overline{\Omega} \backslash \{\mathbf{x}_0\}; \mathbb{R}^n),$$

$$\mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda) = \lambda\mathbf{x} \quad \text{for} \quad \mathbf{x} \in \partial\Omega.$$

(b) For each $\mathbf{x}_0 \in \Omega$

$$\mathbf{u}(\cdot, \mathbf{x}_0, \cdot) \in C^3\left((\overline{\Omega} \backslash \{\mathbf{x}_0\}) \times [\lambda_{\text{crit}}, \infty)\right).$$

(M2) For each $\mathbf{x}_0 \in \Omega$

$$\mathbf{u}(\cdot, \mathbf{x}_0, \lambda) \to \mathbf{u}^h_{\text{crit}}(\cdot) \ \text{ in } \ C^2(\overline{\Omega} \backslash B_\delta(\mathbf{x}_0)) \ \text{ as } \ \lambda \to \lambda^+_{\text{crit}}$$

for any sufficiently small $\delta > 0$, where

$$\mathbf{u}^h_{\text{crit}}(\mathbf{x}) := \lambda_{\text{crit}}\mathbf{x}.$$

(M3) In the case $\Omega = B$, $\mathbf{x}_0 = \mathbf{0}$, the radial minimizer, whose existence is given in Proposition 1.1, is the unique minimizer of E.

Remarks. 1. (M1) and (M2) are known to be satisfied by the radial minimizer given in Proposition 1.1 (see [46]). Further details and implications can be found in [48].

2. It follows from the above hypotheses that each member $\mathbf{u}(\cdot, \mathbf{x}_0, \lambda)$ of our family of minimizers satisfies the equilibrium equations (1.3)–(1.4) in $\Omega \backslash \{\mathbf{x}_0\}$.

Example. Let $n = 3$, $\mathbf{z} = \mathbf{x}_0$, and suppose that $\Omega = B$, the unit ball centered at the origin in $\mathbb{R}^3$. Then it follows from (M1) that each minimizer[16] $\mathbf{u}(\cdot, \mathbf{z})$ satisfies $\mathbf{u}(\mathbf{Q}\mathbf{x}, \mathbf{Q}\mathbf{z}) = \mathbf{Q}\mathbf{u}(\mathbf{x}, \mathbf{z})$ for each $\mathbf{Q} \in \text{SO}(3)$ and $\mathbf{x}, \mathbf{z} \in B$: otherwise, choosing $\tilde{\mathbf{u}}(\mathbf{x}, \mathbf{z}) = \mathbf{Q}^{\text{T}}\mathbf{u}(\mathbf{Q}\mathbf{x}, \mathbf{Q}\mathbf{z})$ we obtain from the assumed frame indifference and isotropy of the material $E(\tilde{\mathbf{u}}) = E(\mathbf{u})$ but $\tilde{\mathbf{u}} \neq \mathbf{u}$ contradicting our assumption of uniqueness. We note that

$$\nabla_{\mathbf{x}}\mathbf{u}(\mathbf{Q}\mathbf{x}, \mathbf{Q}\mathbf{z}) = \mathbf{Q}\nabla_{\mathbf{x}}\mathbf{u}(\mathbf{x}, \mathbf{z})\mathbf{Q}^{\text{T}} \tag{3.1}$$

[16] For ease of exposition we have suppressed the dependence of the minimizer on λ in this example.

and, by the assumed frame indifference and isotropy of W, the energy-momentum tensor (1.5) satisfies

$$\mathbf{M}(\mathbf{QFQ}^{\mathrm{T}}) = \mathbf{QM}(\mathbf{F})\mathbf{Q}^{\mathrm{T}} \tag{3.2}$$

for every $\mathbf{Q} \in \mathrm{SO}(3)$ and $\mathbf{F} \in \mathrm{M}_{+}^{n\times n}$. Therefore, the change of variables $\mathbf{y} = \mathbf{Qx}$ together with (3.1), (3.2), and $\mathbf{n}(\mathbf{Qx}) = \mathbf{Qn}(\mathbf{x})$ yields

$$\begin{aligned}\mathbf{f}(\mathbf{Qz}) &= \int_{\partial B} \mathbf{M}\left(\nabla \mathbf{u}(\mathbf{y}, \mathbf{Qz})\right) \mathbf{n}(\mathbf{y})\, dS_{\mathbf{y}} \\ &= \int_{\partial B} \mathbf{M}\left(\nabla \mathbf{u}(\mathbf{Qx}, \mathbf{Qz})\right) \mathbf{n}(\mathbf{Qx})\, dS_{\mathbf{x}} \\ &= \int_{\partial B} \mathbf{QM}\left(\nabla \mathbf{u}(\mathbf{x}, \mathbf{z})\right) \mathbf{n}(\mathbf{x})\, dS_{\mathbf{x}} = \mathbf{Qf}(\mathbf{z})\end{aligned}$$

for every $\mathbf{Q} \in \mathrm{SO}(3)$ and $\mathbf{z} \in B$. A standard result (see, e.g., [22, p. 238]) then implies that the configurational force is radial: there is a function $\xi : [0, \infty) \to \mathbb{R}$ such that

$$\mathbf{f}(\mathbf{z}) = \xi(|\mathbf{z}|)\mathbf{z}.$$

In particular the configurational force is parallel to $\mathbf{z}$ and its magnitude only depends on the norm of $\mathbf{z}$. We also note that

$$\mathbf{f}(\mathbf{z}) = \nabla_{\mathbf{z}}\eta(|\mathbf{z}|), \qquad \eta(t) := \int_0^t s\xi(s)\, ds,$$

so that the configurational force is the gradient of a potential.

3.2. Expansion of the first variation

We now write each member of our family of equilibria as

$$\mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda) = \lambda\mathbf{x} + \mathbf{w}(\mathbf{x}, \mathbf{x}_0, \lambda). \tag{3.3}$$

Then for each $\mathbf{x}_0 \in \Omega$

$$\begin{aligned} &(i) \quad \mathbf{w}(\mathbf{x}, \mathbf{x}_0, \lambda) = \mathbf{0} \ \text{ for all } \mathbf{x} \in \partial\Omega \text{ and } \lambda > 0, \\ &(ii) \quad \nabla_{\mathbf{x}}\mathbf{w}(\mathbf{x}, \mathbf{x}_0, \lambda) = \frac{\partial \mathbf{w}}{\partial \mathbf{n}} \otimes \mathbf{n} \text{ for all } \mathbf{x} \in \partial\Omega \text{ and } \lambda > 0, \\ &(iii) \quad \mathbf{w}(\mathbf{x}, \mathbf{x}_0, \lambda_{\mathrm{crit}}) \equiv \mathbf{0} \text{ for } \mathbf{x} \in \overline{\Omega}\backslash\{\mathbf{x}_0\}. \end{aligned} \tag{3.4}$$

LEMMA 3.1. *Let (M1)–(M3) hold. Let* $\mathbf{x}_0 \in \Omega$*, let* $\mathbf{m} \in \mathbb{R}^n$ *be a unit vector, and let* $\mathbf{v} \in C_0^1(\Omega; \mathbb{R}^n)$ *satisfy* $\mathbf{v} \equiv \mathbf{m}$ *for all* $\mathbf{x} \in B_\epsilon(\mathbf{x}_0)$ *and some* $\epsilon > 0$*. Then for* $\lambda > \lambda_{\mathrm{crit}}$ *the inner first variation (2.2) is given by*

$$\delta E = \tfrac{1}{2}(\lambda - \lambda_{\mathrm{crit}})^2 \mathbf{m} \cdot \int_{\partial\Omega} \mathbf{n}\nabla_{\mathbf{x}}\dot{\mathbf{w}} : \mathbb{C}[\nabla_{\mathbf{x}}\dot{\mathbf{w}}]\, dS_{\mathbf{x}} + o(|\lambda - \lambda_{\mathrm{crit}}|^2), \tag{3.5}$$

where

$$\dot{\mathbf{w}}(\mathbf{x},\mathbf{x}_0)=\frac{\partial}{\partial\lambda}\mathbf{w}(\mathbf{x},\mathbf{x}_0,\lambda)\Big|_{\lambda=\lambda_{\rm crit}}\qquad \textit{and}\qquad \mathbb{C}=\frac{\partial^2 W}{\partial\mathbf{F}^2}(\lambda_{\rm crit}\mathbf{I})$$

is the elasticity tensor at the (stressed) configuration $\mathbf{u}^h_{\rm crit}(\mathbf{x})\equiv\lambda_{\rm crit}\mathbf{x}$*.*

Proof. We write ∇ to denote $\nabla_{\mathbf{x}}$ and, to simplify notation, suppress the dependence of all functions on the singular point $\mathbf{x}_0$. We first claim that δE satisfies

$$\delta E=-\mathbf{m}\cdot\int_{\partial\Omega}\Phi(\lambda,\mathbf{x})\mathbf{n}(\mathbf{x})\,dS_{\mathbf{x}} \tag{3.6}$$

where

$$\Phi(\lambda,\mathbf{x}):=W(\nabla\mathbf{u}(\mathbf{x},\lambda))-W(\lambda\mathbf{I})-\nabla\mathbf{w}(\mathbf{x},\lambda):\mathbf{S}(\nabla\mathbf{u}(\mathbf{x},\lambda)) \tag{3.7}$$

and $\mathbf{w}(\mathbf{x},\lambda):=\mathbf{u}(\mathbf{x},\lambda)-\lambda\mathbf{x}$. To see this we observe that, by (1.5) and (2.1),

$$\delta E=-\mathbf{m}\cdot\int_{\partial\Omega}\Big[W(\nabla\mathbf{u}(\mathbf{x},\lambda))-(\nabla\mathbf{u}(\mathbf{x},\lambda))^{\rm T}\mathbf{S}(\nabla\mathbf{u}(\mathbf{x},\lambda))\Big]\mathbf{n}\,dS. \tag{3.8}$$

However, by the proof[17] of Lemma 3.2 in [48] (see in particular expression (3.15))

$$\int_{\partial\Omega}\mathbf{S}(\nabla\mathbf{u}(\mathbf{x},\lambda))\mathbf{n}\,dS=-\lim_{\epsilon\to0}\int_{\partial B_\epsilon(\mathbf{x}_0)}\mathbf{S}(\nabla\mathbf{u}(\mathbf{x},\lambda))\mathbf{n}\,dS=\mathbf{0}, \tag{3.9}$$

while the divergence theorem implies

$$\int_{\partial\Omega}W(\lambda\mathbf{I})\mathbf{m}\cdot\mathbf{n}\,dS=0. \tag{3.10}$$

Finally, from (3.4)(ii) we have that $\nabla\mathbf{w}=\frac{\partial\mathbf{w}}{\partial\mathbf{n}}\otimes\mathbf{n}$ on $\partial\Omega$. Therefore for $\mathbf{x}\in\partial\Omega$

$$\mathbf{m}\cdot(\nabla\mathbf{w})^{\rm T}\mathbf{S}\mathbf{n}=\left[\left(\frac{\partial\mathbf{w}}{\partial\mathbf{n}}\otimes\mathbf{n}\right)\mathbf{m}\right]\cdot\mathbf{S}\mathbf{n}=(\mathbf{m}\cdot\mathbf{n})\left(\frac{\partial\mathbf{w}}{\partial\mathbf{n}}\otimes\mathbf{n}\right):\mathbf{S},$$

which together with (3.7)–(3.10) and (3.4)(ii) yields (3.6).

Next, a simple computation[18] shows that $\Phi(\lambda_0,\mathbf{x})=0$, $\Phi_\lambda(\lambda_0,\mathbf{x}))=0$, and

$$\Phi_{\lambda\lambda}(\lambda_0,\mathbf{x},\mathbf{x}_0)=-\nabla\dot{\mathbf{w}}(\mathbf{x},\mathbf{x}_0):\mathbb{C}[\nabla\dot{\mathbf{w}}(\mathbf{x},\mathbf{x}_0)],$$

where $\dot{\mathbf{w}}(\mathbf{x},\mathbf{x}_0)=\mathbf{w}_\lambda(\mathbf{x},\mathbf{x}_0,\lambda_{\rm crit})$. Therefore for each $\mathbf{x}_0\in\Omega$ and $\mathbf{x}\in\overline{\Omega}\setminus\{\mathbf{x}_0\}$

$$\Phi(\lambda,\mathbf{x},\mathbf{x}_0)=-\tfrac{1}{2}(\lambda-\lambda_{\rm crit})^2\nabla\dot{\mathbf{w}}(\mathbf{x},\mathbf{x}_0):\mathbb{C}[\nabla\dot{\mathbf{w}}(\mathbf{x},\mathbf{x}_0)]+o(|\lambda-\lambda_{\rm crit}|^2).$$

Hypothesis (M2) yields $\mathbf{w}(\cdot,\mathbf{x}_0,\lambda)\to\mathbf{0}$ in $C^2(\overline{\Omega}\setminus B_\delta(\mathbf{x}_0))$ as $\lambda\to\lambda^+_{\rm crit}$ for any sufficiently small $\delta>0$. Thus, in particular, the above expansion is uniform for $\mathbf{x}\in\partial\Omega$. Equation (3.5) is now a consequence of the above expansion and (3.6). □

[17] Here (2.5) is crucial.
[18] See [48, p. 205] for details.

LEMMA 3.2. *Let (M1)–(M3) hold then* $\dot{\mathbf{w}}(\mathbf{x},\mathbf{x}_0)$ *satisfies*

$$\mathrm{Div}_{\mathbf{x}}\mathbb{C}[\nabla_{\mathbf{x}}\dot{\mathbf{w}}(\mathbf{x},\mathbf{x}_0)] = \mathbf{0} \ \textit{in}\ \Omega\backslash\{\mathbf{x}_0\}.$$

Proof. From the definition of $\mathbf{w}$ and the remark following the statement of (M1)–(M3) we have that

$$\mathrm{Div}_{\mathbf{x}}\left[\frac{\partial W}{\partial \mathbf{F}}\left(\lambda\mathbf{I} + \nabla_{\mathbf{x}}\mathbf{w}(\mathbf{x},\mathbf{x}_0,\lambda)\right)\right] = \mathbf{0} \quad \text{for} \ \ \mathbf{x}\in\Omega\backslash\{\mathbf{x}_0\}. \tag{3.11}$$

The result now follows on differentiating (3.11) with respect to λ and setting $\lambda = \lambda_{\mathrm{crit}}$. □

4. Asymptotically radial maps

In this section we adopt the ansatz proposed in [48] to evaluate the coefficient of $(\lambda-\lambda_{\mathrm{crit}})^2$ in the expansion of the first variation δE given in Lemma 3.1. The ansatz is the following.

Hypothesis (M4). We assume that the family $\mathbf{u}(\cdot,\mathbf{x}_0,\lambda)$ is **asymptotically radial**, i.e., there are functions $\mu\in C^2(\Omega)$ and $\tilde{\mathbf{v}}:\overline{\Omega}\times\Omega\to\mathbb{R}^n$, with $\tilde{\mathbf{v}}(\cdot,\mathbf{x}_0)\in C^2(\Omega;\mathbb{R}^n)\cap C(\overline{\Omega};\mathbb{R}^n)$ such that for any compact set $S\subset\Omega$, and for $(\lambda-\lambda_{\mathrm{crit}})>0$, the expansion

$$\mathbf{u}(\mathbf{x},\mathbf{x}_0,\lambda) = \lambda_{\mathrm{crit}}\mathbf{x} + (\lambda-\lambda_{\mathrm{crit}})\left[\mu(\mathbf{x}_0)\frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^n} + \tilde{\mathbf{v}}(\mathbf{x},\mathbf{x}_0)\right] + o(|\lambda-\lambda_{\mathrm{crit}}|) \tag{4.1}$$

holds uniformly for $(\mathbf{x},\mathbf{x}_0)\in\overline{\Omega}\backslash B_\delta(\mathbf{x}_0)\times S$ for any $\delta>0$ and sufficiently small.

Remarks. 1. Suppose that we view the minimizing map $\mathbf{u}(\cdot,\mathbf{x}_0,\lambda)$, which is smooth away from the cavity, as a composition of a radial cavitating map followed by a smooth deformation that puts the material at equilibrium. More precisely, let $\mathbf{u}^{(r)}(\cdot,\lambda)$ be the radial minimizer on the unit ball given in Proposition 1.1. Then it is well known [3] that $\mathbf{u}^{(r)}$ has a smooth extension as an equilibrium deformation to all of $\mathbb{R}^n$. Suppose that, for each fixed $\mathbf{x}_0\in\Omega$, we can construct a one-parameter family of smooth deformations $\mathbf{g}(\cdot,\mathbf{x}_0,\lambda)$ that satisfy $\mathbf{u}(\mathbf{x},\mathbf{x}_0,\lambda) = \mathbf{g}(\mathbf{u}^{(r)}(\mathbf{x}-\mathbf{x}_0,\lambda),\mathbf{x}_0,\lambda)$. Then it is shown in [48] that (4.1) is a consequence of the differentiability of this composition with respect to λ at $\lambda=\lambda_{\mathrm{crit}}$.

2. It follows from (3.3) and (4.1) that

$$\dot{\mathbf{w}}(\mathbf{x},\mathbf{x}_0) = \mu(\mathbf{x}_0)\frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^n} + \tilde{\mathbf{v}}(\mathbf{x},\mathbf{x}_0) - \mathbf{x}. \tag{4.2}$$

We define

$$\mathbf{v}(\mathbf{x},\mathbf{x}_0) := \frac{1}{\mu(\mathbf{x}_0)}(\tilde{\mathbf{v}}(\mathbf{x},\mathbf{x}_0)-\mathbf{x}). \tag{4.3}$$

and note that $\mathbf{v}(\mathbf{x},\mathbf{x}_0)$ satisfies the following result which is Corollary 5.3 in [48].

LEMMA 4.1. *Let (M1)–(M4) hold. Then* $\mathbf{v}(\cdot, \mathbf{x}_0)$ *satisfies the linear system of equations*

$$\mathrm{Div}_{\mathbf{x}}\mathbb{C}[\nabla_{\mathbf{x}}\mathbf{v}] = \mathbf{0} \quad \textit{for} \quad \mathbf{x} \in \Omega, \tag{4.4}$$

and the boundary condition

$$\mathbf{v}(\mathbf{x}, \mathbf{x}_0) = -\frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|^n} \quad \textit{for} \quad \mathbf{x} \in \partial\Omega. \tag{4.5}$$

Before proceeding with our next result we note that, as a consequence of the assumed frame indifference and isotropy of W, a standard result (see, e.g., [48, Theorem A.1]) yields constants $a, b, c \in \mathbb{R}$ such that the elasticity tensor $\mathbb{C} = \frac{\partial^2 W}{\partial \mathbf{F}^2}(\lambda_{\mathrm{crit}}\mathbf{I})$ satisfies

$$\mathbb{C}[\mathbf{H}] = a\mathbf{H} + b\mathbf{H}^{\mathrm{T}} + c(\mathrm{trace}\,\mathbf{H})\mathbf{I} \quad \text{for all} \quad \mathbf{H} \in \mathrm{M}^{n\times n}. \tag{4.6}$$

Consequently, (4.4) and the identity $\nabla\mathrm{div}\,\mathbf{v} = \mathrm{Div}[\nabla\mathbf{v}]^{\mathrm{T}}$ imply that the function $\mathbf{v}$ given by (4.3) satisfies

$$\begin{aligned} \mathbf{I} : \mathbb{C}[\nabla\mathbf{v}] &= (a + b + nc)\mathrm{div}\,\mathbf{v}, \\ \mathrm{Div}\,\mathbb{C}[\nabla\mathbf{v}]^{\mathrm{T}} &= b\Delta\mathbf{v} + (a + c)\nabla\mathrm{div}\,\mathbf{v}, \\ \mathbf{0} = \mathrm{Div}\,\mathbb{C}[\nabla\mathbf{v}] &= a\Delta\mathbf{v} + (b + c)\nabla\mathrm{div}\,\mathbf{v}. \end{aligned} \tag{4.7}$$

THEOREM 4.2. *Let (M1)–(M4) hold. Then the coefficient of* $(\lambda - \lambda_{\mathrm{crit}})^2$ *in the expansion of* δE *(see Lemma 3.1) is given by*

$$\tfrac{1}{2}\mathbf{m} \cdot \int_{\partial\Omega} \mathbf{n}\nabla_{\mathbf{x}}\dot{\mathbf{w}} : \mathbb{C}[\nabla_{\mathbf{x}}\dot{\mathbf{w}}]\, dS_{\mathbf{x}} = -\kappa_n\mu(\mathbf{x}_0)^2\mathbf{m} \cdot [\nabla_{\mathbf{x}}(\mathrm{div}_{\mathbf{x}}\mathbf{v}(\mathbf{x}, \mathbf{x}_0))]\Big|_{\mathbf{x}=\mathbf{x}_0},$$

where $\kappa_n = n\omega_n(a + b + c)$ *and* ω_n *is the volume of the unit ball in* $\mathbb{R}^n$ *(*$\omega_2 = \pi$*,* $\omega_3 = 4\pi/3$*).*

Proof. We write once again ∇ to denote $\nabla_{\mathbf{x}}$ and first note that $\dot{\mathbf{w}}$ satisfies the conservation law[19]

$$\mathrm{div}\left((\nabla\dot{\mathbf{w}} : \mathbb{C}[\nabla\dot{\mathbf{w}}])\mathbf{m} - 2m^{\alpha}\mathbb{C}[\nabla\dot{\mathbf{w}},_{\alpha}]^{\mathrm{T}}\dot{\mathbf{w}}\right) = \mathbf{0} \quad \text{in} \quad \Omega\backslash\{\mathbf{x}_0\}. \tag{4.8}$$

To see this we observe that, by Lemma 3.2, $\mathbf{0} = (\mathrm{Div}\,\mathbb{C}[\nabla\dot{\mathbf{w}}]),_{\alpha} = \mathrm{Div}\,\mathbb{C}[\nabla\dot{\mathbf{w}},_{\alpha}]$. Consequently

$$\mathrm{div}\left(\mathbb{C}[\nabla\dot{\mathbf{w}},_{\alpha}]^{\mathrm{T}}\dot{\mathbf{w}}\right) = \nabla\dot{\mathbf{w}} : \mathbb{C}[\nabla\dot{\mathbf{w}},_{\alpha}]$$

and hence, if we multiply by m^{α} and sum over the index α,

$$\mathrm{div}\left(m^{\alpha}\mathbb{C}[\nabla\dot{\mathbf{w}},_{\alpha}]^{\mathrm{T}}\dot{\mathbf{w}}\right) = m^{\alpha}\nabla\dot{\mathbf{w}} : \mathbb{C}[\nabla\dot{\mathbf{w}},_{\alpha}].$$

[19] Equivalently, $m^{\alpha}\mathbb{C}[\nabla\dot{\mathbf{w}},_{\alpha}] = \mathbb{C}[\nabla((\nabla\dot{\mathbf{w}})\mathbf{m})]$. Recall that we use the convention of summation over repeated indices.

However, in view of the symmetry[20] of $\mathbb{C}$

$$\operatorname{div}((\nabla\dot{\mathbf{w}} : \mathbb{C}[\nabla\dot{\mathbf{w}}])\mathbf{m}) = \mathbf{m}\cdot\nabla(\nabla\dot{\mathbf{w}} : \mathbb{C}[\nabla\dot{\mathbf{w}}]) = 2m^\alpha\nabla\dot{\mathbf{w}} : \mathbb{C}[\nabla\dot{\mathbf{w}},_\alpha],$$

which together with the previous equation yields (4.8).

We next note that $\dot{\mathbf{w}} = \mathbf{0}$ on $\partial\Omega$ and so the coefficient of $(\lambda-\lambda_{\rm crit})^2$ in the expansion (3.5) of δE is given by

$$\begin{aligned} Q(\dot{\mathbf{w}}) &= +\tfrac{1}{2}\int_{\partial\Omega}(\mathbf{m}\cdot\mathbf{n})\nabla\dot{\mathbf{w}} : \mathbb{C}[\nabla\dot{\mathbf{w}}]\,dS \qquad (4.9)\\ &= +\tfrac{1}{2}\int_{\partial\Omega}((\mathbf{m}\cdot\mathbf{n})\nabla\dot{\mathbf{w}} : \mathbb{C}[\nabla\dot{\mathbf{w}}] - 2m^\alpha\dot{\mathbf{w}}\cdot\mathbb{C}[\nabla\dot{\mathbf{w}},_\alpha]\mathbf{n})\,dS\\ &= -\tfrac{1}{2}\int_{\partial B_\epsilon}((\mathbf{m}\cdot\mathbf{n})\nabla\dot{\mathbf{w}} : \mathbb{C}[\nabla\dot{\mathbf{w}}] - 2m^\alpha\dot{\mathbf{w}}\cdot\mathbb{C}[\nabla\dot{\mathbf{w}},_\alpha]\mathbf{n})\,dS, \qquad (4.10)\end{aligned}$$

where $B_\epsilon = B_\epsilon(\mathbf{x}_0)$ is the (open) ball of radius ϵ centered at $\mathbf{x}_0$ and we have made use of (4.8) and the divergence theorem to obtain the last equality.

In view of (4.2) and (4.3) we next set

$$\dot{\mathbf{w}} = \mu[\mathbf{v}+\mathbf{p}], \qquad \mathbf{p}(\mathbf{x}) := \frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^n} \qquad (4.11)$$

in (4.10), expand in terms of $\mathbf{v}$ and $\mathbf{p}$, and evaluate the integral (4.10) in the limit $\epsilon\to 0^+$. In evaluating the limiting value of this integral it is clear that the terms that are quadratic in $\mathbf{v}$ converge to zero as $\epsilon\to 0^+$ by the smoothness of $\mathbf{v}$. Moreover, the most singular terms in the expansion of (4.10) are those that are quadratic in $\mathbf{p}$ and its derivatives and are given by

$$-\tfrac{1}{2}\mu^2\int_{\partial B_\epsilon}\Big[(\mathbf{m}\cdot\mathbf{n})\nabla\mathbf{p} : \mathbb{C}[\nabla\mathbf{p}] - 2m^\alpha\mathbf{p}\cdot\mathbb{C}[\nabla\mathbf{p},_\alpha]\mathbf{n}\Big]\,dS. \qquad (4.12)$$

For $\mathbf{x}\in\partial B_\epsilon(\mathbf{x}_0)$ the function $\mathbf{p}$ and its derivatives satisfy

$$\mathbf{p} = O\left(\epsilon^{1-n}\right), \quad \nabla\mathbf{p} = O\left(\epsilon^{-n}\right), \quad \nabla\mathbf{p},_\alpha = O\left(\epsilon^{-n-1}\right), \qquad (4.13)$$

as $\epsilon\to 0^+$. From these equations it is clear that the singular integral terms in (4.12) are each of order ϵ^{-2n} and hence their integrals over $\partial B_\epsilon(\mathbf{x}_0)$ are each of order ϵ^{-n-1} as $\epsilon\to 0^+$. The sum of these terms contribute nothing to the integral in the limit $\epsilon\to 0^+$ as can be directly verified or by observing that the original integral (4.9) is clearly finite and independent of ϵ.

Thus it remains to use (4.11) and evaluate the terms in the expansion of (4.10) that include both $\mathbf{p}$ and $\mathbf{v}$. On noting that, for the domain $\Omega\setminus B_\epsilon$, the (outward) pointing normal $\mathbf{n}$ on ∂B_ϵ is given by $\mathbf{n} = -\frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|}$, and the fact that $\mathbb{C}$ is symmetric it follows that the cross terms in question are given by

$$\mu^2\int_{\partial B_\epsilon}\Big[(\mathbf{m}\cdot\hat{\mathbf{n}})\nabla\mathbf{p} : \mathbb{C}[\nabla\mathbf{v}] - m^\alpha\mathbf{v}\cdot\mathbb{C}[\nabla\mathbf{p},_\alpha]\hat{\mathbf{n}} - m^\alpha\mathbf{p}\cdot\mathbb{C}[\nabla\mathbf{v},_\alpha]\hat{\mathbf{n}}\Big]dS, \qquad (4.14)$$

[20] The second derivative of W is *symmetric*: $\mathbf{A} : \mathbb{C}[\mathbf{B}] = \mathbf{B} : \mathbb{C}[\mathbf{A}]$ for any $\mathbf{A},\mathbf{B}\in\mathrm{M}^{n\times n}$.

where

$$\hat{\mathbf{n}} = -\mathbf{n} = \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|} \tag{4.15}$$

is the *outward* unit normal to the boundary of the ball $B_\epsilon(\mathbf{x}_0)$.

We next calculate the contributions from each of the three terms listed in (4.14).

Term I. We first, note that

$$\nabla \mathbf{p}(\mathbf{x}) = \frac{1}{|\mathbf{x} - \mathbf{x}_0|^n} \left[\mathbf{I} - n \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|} \otimes \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|} \right] \tag{4.16}$$

and, in view of (4.15),

$$([(\mathbf{x} - \mathbf{x}_0) \otimes (\mathbf{x} - \mathbf{x}_0)] : \mathbf{C})\hat{\mathbf{n}} = \Big[(\mathbf{x} - \mathbf{x}_0) \otimes \mathbf{C}^{\mathrm{T}} (\mathbf{x} - \mathbf{x}_0) \Big] \hat{\mathbf{n}},$$

where $\mathbf{C} = \mathbf{C}(\mathbf{x}, \mathbf{x}_0) := \mathbb{C}[\nabla \mathbf{v}(\mathbf{x}, \mathbf{x}_0)]$. Thus, by (4.16), the divergence theorem, and the fact that $|\mathbf{x} - \mathbf{x}_0| = \epsilon$ for $\mathbf{x} \in \partial B_\epsilon$, the first integral in (4.14) is equal to the inner product of $\mu^2(\mathbf{x}_0)\mathbf{m}$ with the vector

$$\begin{aligned} & \frac{1}{\epsilon^n} \int_{\partial B_\epsilon} (\mathbf{C} : \mathbf{I})\, \hat{\mathbf{n}}\, dS - \frac{n}{\epsilon^{n+2}} \int_{\partial B_\epsilon} \Big[(\mathbf{x} - \mathbf{x}_0) \otimes \mathbf{C}^{\mathrm{T}} (\mathbf{x} - \mathbf{x}_0) \Big] \hat{\mathbf{n}}\, dS \\ = \; & \frac{1}{\epsilon^n} \int_{B_\epsilon} \nabla (\mathbf{C} : \mathbf{I})\, d\mathbf{x} - \frac{n}{\epsilon^{n+2}} \int_{B_\epsilon} \mathrm{Div} \Big[(\mathbf{x} - \mathbf{x}_0) \otimes \mathbf{C}^{\mathrm{T}} (\mathbf{x} - \mathbf{x}_0) \Big] d\mathbf{x}. \end{aligned} \tag{4.17}$$

We note that $\mathrm{Div}_{\mathbf{x}} \mathbf{C} = \mathbf{0}$ and consequently

$$\mathrm{Div} \Big[(\mathbf{x} - \mathbf{x}_0) \otimes \mathbf{C}^{\mathrm{T}} (\mathbf{x} - \mathbf{x}_0) \Big] = \mathbf{C}^{\mathrm{T}} (\mathbf{x} - \mathbf{x}_0) + (\mathbf{x} - \mathbf{x}_0) \mathbf{C} : \mathbf{I}. \tag{4.18}$$

Finally, we substitute and (4.18) into (4.17) and then let $\epsilon \to 0^+$ to conclude with the aid of Proposition A.1 and $(4.7)_{1,2}$ that the first integral in (4.14) is equal to

$$\begin{aligned} & \frac{\mu^2 \omega_n}{n+2} \mathbf{m} \cdot \Big[2\nabla \left(\mathbb{C}[\nabla \mathbf{v}] : \mathbf{I} \right) - n \mathrm{Div}\, \mathbb{C}[\nabla \mathbf{v}]^{\mathrm{T}} \Big] \\ = \; & \frac{\mu^2 \omega_n}{n+2} \mathbf{m} \cdot \left[(2(a+b) + n(c-a))\, \nabla \mathrm{div}\, \mathbf{v}(\mathbf{x}_0, \mathbf{x}_0) - nb \Delta \mathbf{v}(\mathbf{x}_0, \mathbf{x}_0) \right]. \end{aligned} \tag{4.19}$$

Term II. First note that $\mathbf{p} = \nabla \zeta$, where ζ is the fundamental solution of Laplace's equation. Thus $\nabla \mathbf{p},_\alpha$ is a symmetric matrix and $\mathrm{trace} \nabla \mathbf{p},_\alpha = (\mathrm{div}\, \mathbf{p}),_\alpha = (\Delta \zeta),_\alpha = 0$. Thus $\mathbb{C}[\nabla \mathbf{p},_\alpha] = (a+b) \nabla \mathbf{p},_\alpha$ by (4.6).

We next differentiate (4.16) to get

$$\begin{aligned} p,^{\gamma}_{\alpha\beta}(\mathbf{x}) = \frac{-n}{|\mathbf{x} - \mathbf{x}_0|^{n+2}} \bigg(& \delta^\alpha_\beta (x^\gamma - x^\gamma_0) + \delta^\gamma_\beta (x^\alpha - x^\alpha_0) + \delta^\alpha_\gamma (x^\beta - x^\beta_0) \\ & - (n+2) \frac{(x^\alpha - x^\alpha_0)(x^\beta - x^\beta_0)(x^\gamma - x^\gamma_0)}{|\mathbf{x} - \mathbf{x}_0|^2} \bigg). \end{aligned}$$

Therefore for $\mathbf{x} \in \partial B_\epsilon$ (so that $|\mathbf{x}-\mathbf{x}_0| = \epsilon$) we find with the aid of (4.15) that

$$\begin{aligned} m^\alpha \mathbf{v}\cdot\mathbb{C}[\nabla \mathbf{p},_\alpha]\hat{\mathbf{n}} &= \frac{(a+b)n}{\epsilon^{n+1}}\left[n(\mathbf{v}\cdot\hat{\mathbf{n}})(\mathbf{m}\cdot\hat{\mathbf{n}}) - (\mathbf{m}\cdot\mathbf{v})\right] \\ &= \frac{(a+b)n}{\epsilon^{n+2}}\mathbf{m}\cdot\left[n\left((\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{v}\right)\mathbf{I} - \mathbf{v}\otimes(\mathbf{x}-\mathbf{x}_0)\right]\hat{\mathbf{n}}. \end{aligned}$$

Consequently, by the divergence theorem and the identity

$$\operatorname{Div}\left[n\left((\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{v}\right)\mathbf{I} - \mathbf{v}\otimes(\mathbf{x}-\mathbf{x}_0)\right] = n[\nabla\mathbf{v}]^{\mathrm{T}}(\mathbf{x}-\mathbf{x}_0) - [\nabla\mathbf{v}](\mathbf{x}-\mathbf{x}_0),$$

the second term in (4.14) is equal to

$$\begin{aligned} &\frac{\mu^2 n(a+b)}{\epsilon^{n+2}}\mathbf{m}\cdot\int_{B_\epsilon}\Big([\nabla\mathbf{v}](\mathbf{x}-\mathbf{x}_0) - n[\nabla\mathbf{v}]^{\mathrm{T}}(\mathbf{x}-\mathbf{x}_0)\Big)\,d\mathbf{x} \\ \to\ &\frac{\mu^2 n\omega_n(a+b)}{n+2}\mathbf{m}\cdot\left[\Delta\mathbf{v}(\mathbf{x}_0,\mathbf{x}_0) - n\nabla\operatorname{div}\mathbf{v}(\mathbf{x}_0,\mathbf{x}_0)\right] \end{aligned} \tag{4.20}$$

as $\epsilon \to 0^+$, in view of Proposition A.1 and the identity $\operatorname{Div}[\nabla\mathbf{v}]^{\mathrm{T}} = \nabla\operatorname{div}\mathbf{v}$.

Term III. We first note that, by (4.4), $\mathbf{0} = (\operatorname{Div}(\mathbb{C}[\nabla\mathbf{v}]),_\alpha = \operatorname{Div}\mathbb{C}[\nabla\mathbf{v},_\alpha]$ and hence

$$\operatorname{div}\Big(\mathbb{C}[\nabla\mathbf{v},_\alpha]^{\mathrm{T}}(\mathbf{x}-\mathbf{x}_0)\Big) = \mathbf{I} : \mathbb{C}[\nabla\mathbf{v},_\alpha] \quad \text{in } \Omega. \tag{4.21}$$

Therefore, by $(4.11)_2$, (4.21), and the divergence theorem we find that

$$\begin{aligned} \mu^2\int_{\partial B_\epsilon} m^\alpha \mathbf{p}\cdot\mathbb{C}[\nabla\mathbf{v},_\alpha]\hat{\mathbf{n}}\,dS &= \frac{\mu^2}{\epsilon^n}\int_{\partial B_\epsilon} m^\alpha(\mathbf{x}-\mathbf{x}_0)\cdot\mathbb{C}[\nabla\mathbf{v},_\alpha]\hat{\mathbf{n}}\,dS \\ &= \frac{\mu^2}{\epsilon^n}\int_{B_\epsilon} m^\alpha \mathbf{I} : \mathbb{C}[\nabla\mathbf{v},_\alpha]\,d\mathbf{x} \\ &\to \mu^2\omega_n m^\alpha \mathbf{I} : \mathbb{C}[\nabla\mathbf{v},_\alpha(\mathbf{x}_0,\mathbf{x}_0)] \\ &= \mu^2\omega_n(a+b+nc)\mathbf{m}\cdot\nabla\operatorname{div}\mathbf{v}(\mathbf{x}_0,\mathbf{x}_0) \end{aligned} \tag{4.22}$$

as $\epsilon \to 0^+$, where we have once again made use of $(4.7)_1$.

Finally, we subtract (4.22) from the sum of (4.19) and (4.20) and use $(4.7)_3$ to obtain the desired result. □

5. An Example

For the remainder of the paper we take $n = 3$ and $\Omega = B$, the unit ball in $\mathbb{R}^3$. We also assume that $\mathbb{C}$ is strongly elliptic so that (see, e.g., [48, Theorem A.1]) $a > 0$

and $a+b+c>0$. The next result gives a formula for $\mathbf{v}(\mathbf{x},\mathbf{x}_0)$ from which we can explicitly calculate $\nabla_{\mathbf{x}}\mathrm{div}_{\mathbf{x}}\mathbf{v}(\mathbf{x},\mathbf{x}_0)$ and hence evaluate the coefficient in the above expansion. The formula is due originally to Lord Kelvin [50]) and can also be found in [36]. To begin, choose a rectangular coordinate system (x_1,x_2,x_3) on $\mathbb{R}^3$ so that $\mathbf{x}_0\in B$ satisfies $\mathbf{x}_0=(0,0,|\mathbf{x}_0|)$ and define spherical coordinates ρ, ϕ, and θ by

$$x_1=\rho(\cos\theta)(\sin\phi),\quad x_2=\rho(\sin\theta)(\sin\phi),\quad x_3=\rho(\cos\phi).$$

PROPOSITION 5.1. *([48, Proposition 7.1]) Let $\mathbb{C}$ be strongly elliptic. Then the unique solution of (4.4)–(4.5) is given by*

$$\mathbf{v}(\mathbf{x},\mathbf{x}_0)=\sum_{k=1}^{\infty}\mathbf{v}^{(k)}(\mathbf{x},\mathbf{x}_0)+(1-\rho^2)\sum_{k=2}^{\infty}\tau_k\nabla_{\mathbf{x}}(\mathrm{div}_{\mathbf{x}}\mathbf{v}^{(k)}(\mathbf{x},\mathbf{x}_0)),\tag{5.1}$$

where

$$\mathbf{v}^{(k)}=-\rho^k|\mathbf{x}_0|^{k-1}(P_k'(\cos\phi)\cos\theta\sin\phi,P_k'(\cos\phi)\sin\theta\sin\phi,kP_k(\cos\phi)),\tag{5.2}$$

P_k denotes the k^{th}-Legendre polynomial, and

$$\tau_k:=\frac{\frac{1}{2}(b+c)}{(2a+b+c)k-(a+b+c)}.\tag{5.3}$$

Remark. Strong ellipticity is used to obtain the uniqueness of solutions to (4.4)–(4.5).

The next result uses the above formula for $\mathbf{v}$ to explicitly calculate $\nabla\mathrm{div}\,\mathbf{v}$.

THEOREM 5.2. *Let $\mathbf{v}(\mathbf{x},\mathbf{x}_0)$ satisfy (5.1)–(5.3) then*

$$[\nabla_{\mathbf{x}}(\mathrm{div}_{\mathbf{x}}\mathbf{v})]|_{\mathbf{x}=\mathbf{x}_0}=-a\left[\sum_{k=2}^{\infty}|\mathbf{x}_0|^{2k-4}\frac{k(k-1)(2k-1)(2k+1)}{(2a+b+c)k-(a+b+c)}\right]\mathbf{x}_0.\tag{5.4}$$

Proof. In order to compute the required derivatives of $\mathbf{v}$ we first take the divergence of (5.2) and make use of the identities $(1-t^2)P_k''(t)=2tP_k'(t)-k(k+1)P_k(t)$ and $(1-t^2)P_k'(t)+ktP_k(t)=P_{k-1}(t)$ to get

$$\mathrm{div}_{\mathbf{x}}\mathbf{v}^{(k)}=-k(2k+1)\rho^{k-1}|\mathbf{x}_0|^{k-1}P_{k-1}'(\cos\phi)$$

and consequently $\nabla(\mathrm{div}\,\mathbf{v}^{(k)})$ is equal to

$$-k(2k+1)\rho^{k-2}|\mathbf{x}_0|^{k-1}\begin{bmatrix}\cos\theta\sin\phi\left[(k-1)P_{k-1}-(\cos\phi)P_{k-1}'\right]\\ \sin\theta\sin\phi\left[(k-1)P_{k-1}-(\cos\phi)P_{k-1}'\right]\\ (k-1)(\cos\phi)P_{k-1}+(\sin^2\phi)P_{k-1}'\end{bmatrix}.\tag{5.5}$$

We now consider two cases: (I) $b+c=0$ and (II) $b+c\neq 0$. If $b+c=0$ then $\tau_k=0$ and hence if we let $\mathbf{x}=\mathbf{x}_0=(0,0,|\mathbf{x}_0|)$ and consequently $\phi=0$, $\rho=|\mathbf{x}_0|$, we find that (5.1), (5.5), and the fact that $P_{k-1}(1)=1$ imply

$$[\nabla_{\mathbf{x}}(\mathrm{div}_{\mathbf{x}}\mathbf{v}^{(k)})]|_{\mathbf{x}=\mathbf{x}_0}=-k(2k+1)|\mathbf{x}_0|^{2k-3}(0,0,(k-1)), \tag{5.6}$$

and therefore

$$[\nabla_{\mathbf{x}}\left(\mathrm{div}_{\mathbf{x}}\mathbf{v}\right)]\left.\right|_{\mathbf{x}=\mathbf{x}_0}=\sum_{k=1}^{\infty}-k(2k+1)(k-1)|\mathbf{x}_0|^{2k-3}(0,0,1),$$

which is (5.4) with $b+c=0$.

If $b+c\neq 0$ then by $(4.7)_3$

$$\nabla(\mathrm{div}\,\mathbf{v})=\frac{-a}{b+c}\Delta\mathbf{v}. \tag{5.7}$$

We next compute $\Delta_{\mathbf{x}}\mathbf{v}$ from the infinite series (5.1)–(5.2).

First, $\Delta\mathbf{v}^{(k)}=0$ and so does not contribute. Next,

$$\nabla[(1-\rho^2)\mathbf{w}^{(k)}]=(1-\rho^2)\nabla\mathbf{w}^{(k)}+\mathbf{w}^{(k)}\otimes\nabla(1-\rho^2),\quad \mathbf{w}^{(k)}:=\nabla(\mathrm{div}\,\mathbf{v}^{(k)}),$$

and thus if we take the divergence of both sides

$$\Delta[(1-\rho^2)\mathbf{w}^{(k)}]=(1-\rho^2)\Delta\mathbf{w}^{(k)}+2(\nabla\mathbf{w}^{(k)})[\nabla(1-\rho^2)]+(\Delta(1-\rho^2))\mathbf{w}^{(k)}.$$

However,

$$\Delta\mathbf{w}^{(k)}=\mathbf{0},\quad \nabla(1-\rho^2)=-2(x_1,x_2,x_3),\quad \Delta(1-\rho^2)=-6,$$

and consequently

$$\Delta[(1-\rho^2)\mathbf{w}^{(k)}]=-4(\nabla\mathbf{w}^{(k)})[(x_1,x_2,x_3)]-6\mathbf{w}^{(k)}. \tag{5.8}$$

Now

$$(\nabla\mathbf{w}^{(k)})[(x_1,x_2,x_3)]=\rho\frac{\partial}{\partial\rho}\mathbf{w}^{(k)}=\rho\frac{\partial}{\partial\rho}\nabla\mathrm{div}\,\mathbf{v}^{(k)}$$

and so if we take the partial derivative of (5.5) with respect to ρ, multiply by ρ, and set $\mathbf{x}=\mathbf{x}_0=(0,0,|\mathbf{x}_0|)$ ($\phi=0$, $\rho=|\mathbf{x}_0|$, $P_{k-1}(1)=1$) we find that

$$(\nabla\mathbf{w}^{(k)})[(x_1,x_2,x_3)]\Big|_{\mathbf{x}=\mathbf{x}_0}=-k(2k+1)(k-2)|\mathbf{x}_0|^{2k-3}(0,0,(k-1)),$$

which together with (5.1), (5.6), and (5.8) yields

$$\Delta_{\mathbf{x}}\mathbf{v}(\mathbf{x},\mathbf{x}_0)|_{\mathbf{x}=\mathbf{x}_0}=\sum_{k=1}^{\infty}\tau_k k(2k+1)(k-1)[4(k-2)+6]|\mathbf{x}_0|^{2k-3}(0,0,1). \tag{5.9}$$

The desired result now follows from (5.3), (5.7), and (5.9). □

6. Conclusions

Our main result is the following.

THEOREM 6.1. *Let (M1)–(M4) hold. Suppose* $\mathbb{C}$ *is strongly elliptic,* $n = 3$, *and* $\Omega = B$, *the unit ball in* $\mathbb{R}^3$. *Then the configurational force on* $\mathbf{x}_0$ *satisfies*

$$\mathbf{f}(\mathbf{x}_0, \lambda) = -\Psi(|\mathbf{x}_0|, \lambda)\mathbf{x}_0, \quad \Psi(t, \lambda) = (\lambda - \lambda_{\text{crit}})^2\psi(t) + o(|\lambda - \lambda_{\text{crit}}|^2),$$

where $\psi > 0$. *Consequently if* $\mathbf{x}_0 \neq \mathbf{0}$ *then* $\mathbf{f}(\mathbf{x}_0, \lambda) \neq \mathbf{0}$ *for* $\lambda - \lambda_{\text{crit}} > 0$ *and sufficiently small.*

Proof. In light of the example at the end of Section 3.1, $\mathbf{f}(\mathbf{x}_0, \lambda)$ is parallel to $\mathbf{x}_0$ and its length only depends on the norm of $\mathbf{x}_0$ (and λ). Thus

$$\mathbf{f}(\mathbf{x}_0, \lambda) = -\Psi(|\mathbf{x}_0|, \lambda)\mathbf{x}_0,$$

for an appropriate scalar-valued function Ψ. However, by (2.2), Lemma 3.1, Theorem 4.2, and Theorem 5.2

$$-\mathbf{f}(\mathbf{x}_0, \lambda) \cdot \frac{\mathbf{x}_0}{|\mathbf{x}_0|} = \delta E = (\lambda - \lambda_{\text{crit}})^2\psi(|\mathbf{x}_0|)|\mathbf{x}_0| + o(|\lambda - \lambda_{\text{crit}}|^2),$$

where

$$\psi(s) := \kappa_n \mu(s)^2 \left[\sum_{k=2}^{\infty} |\mathbf{x}_0|^{2k-4} \frac{ak(k-1)(2k-1)(2k+1)}{(2a+b+c)k - (a+b+c)}\right].$$

Finally, the strong ellipticity of $\mathbb{C}$ ($a > 0$, $a + b + c > 0$) yields the indicated sign for ψ. □

Consequently, the center is the only point in the ball where cavitation does not generate configurational forces and so it is the only possible optimal cavitation point.

COROLLARY 6.2. *Let the hypotheses of Theorem 6.1 be satisfied. Then for every* $\mathbf{x}_0 \in B$, *with* $\mathbf{x}_0 \neq \mathbf{0}$, *and every sufficiently small* $\lambda - \lambda_{\text{crit}} > 0$ *there is a* $\mathbf{z} \in B$ *such that*

$$E(\mathbf{u}(\cdot, \mathbf{z}, \lambda)) < E(\mathbf{u}(\cdot, \mathbf{x}_0, \lambda)),$$

where E *is the total elastic energy given by (1.2) and* $\mathbf{u}(\cdot, \mathbf{y}, \lambda)$ *is the minimizer of (1.2) among all deformations that satisfy (1.1) and open at most a single new hole at* $\mathbf{y} \in B$.

Proof. Fix $\mathbf{x}_0 \in B$ with $\mathbf{x}_0 \neq \mathbf{0}$. Then in view of Theorem 6.1 and (2.2)

$$E(\mathbf{u}_t(\cdot, \mathbf{x}_0, \lambda)) < E(\mathbf{u}(\cdot, \mathbf{x}_0, \lambda)), \tag{6.1}$$

whenever $t > 0$ and $\lambda - \lambda_{\text{crit}} > 0$ are sufficiently small. However, $\mathbf{u}_t(\cdot, \mathbf{x}_0, \lambda)$ creates a cavity at the point $\mathbf{x}_0 - t\mathbf{m}$, $\mathbf{m} := \mathbf{x}_0/|\mathbf{x}_0|$ and thus

$$E(\mathbf{u}(\cdot, \mathbf{x}_0 - t\mathbf{m}, \lambda)) \le E(\mathbf{u}_t(\cdot, \mathbf{x}_0, \lambda)), \tag{6.2}$$

since $\mathbf{u}(\cdot, \mathbf{x}_0 - t\mathbf{m}, \lambda)$ is the minimizer of E among deformations that create a cavity at $\mathbf{x}_0 - t\mathbf{m}$. The desired result now follows from (6.1) and (6.2). □

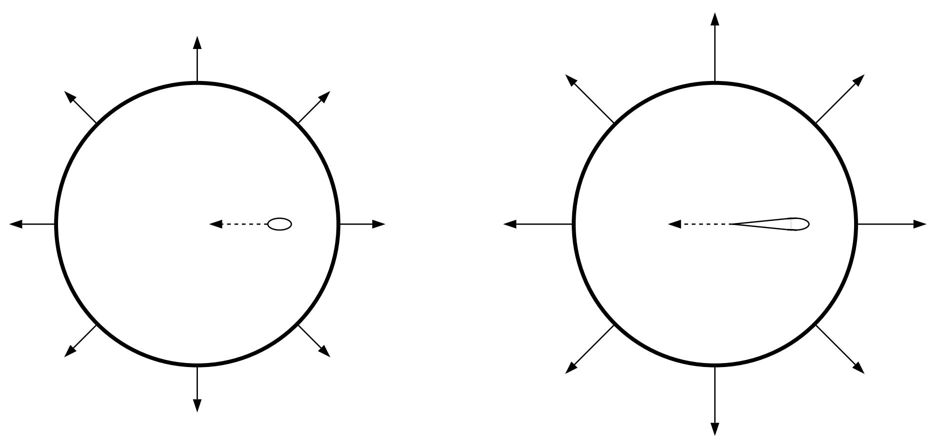

Figure 1. Cavitation followed by Fracture.

Finally, Figure 1 shows our speculation as to how fracture might follow cavitation, in 2-dimensions, as mentioned in the Introduction. The imposed displacements on the boundary (solid arrows) first induce a cavity to form in the illustration on the left. This cavity has a configurational force (dotted-arrow) in the direction of the center. In a purely elastic theory there must necessarily be no other precursors closer to the origin. If additional energy (e.g., surface energy) is instead attributed to each precursor then cavitation could occur first at this point provided a single large microvoid, with small added energy inhibiting cavitation, were located there. We hypothesize that further boundary displacement generates a crack that propagates toward the center of the material in the illustration on the right. The analysis of Rice [42, 43] (see also [9, 10, 16, 17, 23, 30]) may be particularly relevant here since δE is the J-integral in this problem.

Appendix

In this section we prove the following result

PROPOSITION A.1. *Let* $\mathbf{G} : \Omega \to \mathrm{M}^{n\times n}$ *be continuously differentiable in a neighborhood of* $\mathbf{x}_0$*. Then*

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon^{n+2}} \int_{B_\epsilon(\mathbf{x}_0)} \mathbf{G}(\mathbf{x})(\mathbf{x}-\mathbf{x}_0)\,d\mathbf{x} = \frac{\omega_n}{n+2}\mathrm{Div}\,\mathbf{G}(\mathbf{x}_0),$$

where ω_n *is the volume of the unit ball in* $\mathbb{R}^n$*.*

Proof. We first note that $(\mathbf{x}-\mathbf{x}_0) = \frac{1}{2}\nabla_{\mathbf{x}}|\mathbf{x}-\mathbf{x}_0|^2$ and hence

$$2\mathbf{G}(\mathbf{x})(\mathbf{x}-\mathbf{x}_0) = \mathrm{Div}\left(|\mathbf{x}-\mathbf{x}_0|^2\mathbf{G}(\mathbf{x})\right) - |\mathbf{x}-\mathbf{x}_0|^2\mathrm{Div}\,\mathbf{G}(\mathbf{x}).$$

Therefore, by the divergence theorem

$$\int_{B_\epsilon} \left[2\mathbf{G}(\mathbf{x})(\mathbf{x}-\mathbf{x}_0) + |\mathbf{x}-\mathbf{x}_0|^2\mathrm{Div}\,\mathbf{G}(\mathbf{x})\right] d\mathbf{x}$$

$$= \int_{\partial B_\epsilon} |\mathbf{x}-\mathbf{x}_0|^2\mathbf{G}\mathbf{n}\,dS = \epsilon^2\int_{\partial B_\epsilon} \mathbf{G}\mathbf{n}\,dS = \epsilon^2\int_{B_\epsilon} \mathrm{Div}\,\mathbf{G}\,d\mathbf{x},$$

where $\mathbf{n}$ is the outward unit normal to the boundary of $B_\epsilon := B_\epsilon(\mathbf{x}_0)$: the ball of radius ϵ centered at $\mathbf{x}_0$. Consequently,

$$\frac{2}{\epsilon^{n+2}}\int_{B_\epsilon} \mathbf{G}(\mathbf{x})(\mathbf{x}-\mathbf{x}_0)\,d\mathbf{x} \;=\; \frac{1}{\epsilon^n}\int_{B_\epsilon}\mathrm{Div}\,\mathbf{G}\,d\mathbf{x} - \frac{1}{\epsilon^{n+2}}\int_{B_\epsilon}|\mathbf{x}-\mathbf{x}_0|^2\mathrm{Div}\,\mathbf{G}\,d\mathbf{x}$$

and the desired result now follows from the continuity of $\mathrm{Div}\,\mathbf{G}$ at $\mathbf{x}_0$ and the Lebesgue differentiation theorem for Lebesgue measure $d\mathbf{x}$ as well as the measure $dm_{\mathbf{x}} := |\mathbf{x}-\mathbf{x}_0|^p d\mathbf{x}$ (for $p > -n$, $m(B_\epsilon(\mathbf{x}_0)) = n\omega_n\epsilon^{n+p}/(n+p)$). □

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