Construction of infinitely many singular weak solutions to the equations of nonlinear elasticity

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Abstract

Radial deformations of a ball composed of a nonlinear elastic material and corresponding to cavitation have been much studied (see [7]). In this note we use rescalings to show that each such deformation can be used to construct infinitely many non-symmetric, singular weak solutions of the equations of nonlinear elasticity for the same displacement boundary-value problem. Surprisingly, this property appears to have been unnoticed in the literature to date.

Let $\Omega \subset \mathbb{R}^n$ denote the reference state of a nonlinearly elastic body, let $p \geq 1$, and let

$$\mathbf{u} : \Omega \to \mathbb{R}^n, \quad \mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^n), \quad \det \nabla \mathbf{u} > 0 \quad \text{a.e.},$$

denote a deformation of the body. In hyperelasticity we associate with each deformation the energy

$$E(\mathbf{u}) = \int_\Omega W(\nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x}, \tag{1}$$

where $W : \mathrm{M}^{n\times n}_{+} \to \mathbb{R}^{+}$ is the stored-energy function and $\mathrm{M}^{n\times n}_{+}$ denotes the $n \times n$ matrices with positive determinant (see e.g. [4]). The equilibrium equations of nonlinear elasticity are the Euler-Lagrange equations for (1) and can take a number of forms. For the purposes of this paper we note the following two forms:

$$\frac{\partial}{\partial x^{\alpha}}\left[W(\nabla\mathbf{u})\,\delta^{\beta}_{\alpha} - u,^{k}_{\beta}\,\frac{\partial W}{\partial F^{k}_{\alpha}}(\nabla\mathbf{u})\right] = 0, \quad \beta = 1, 2, \ldots, n, \tag{2}$$

and

$$\frac{\partial}{\partial x^{\alpha}}\left[\frac{\partial W}{\partial F^{i}_{\alpha}}(\nabla\mathbf{u})\right] = 0, \quad i = 1, 2, \ldots, n, \tag{3}$$

where we have used the summation convention for repeated indices. (Equation (2) is sometimes referred to as the energy-momentum form of the equilibrium equations.)

Suppose further that the deformation $\mathbf{u}$ satisfies the affine boundary condition

$$\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x} \qquad \text{for } \mathbf{x} \in \partial\Omega,$$

where $\mathbf{A} \in \mathrm{M}^{n\times n}_{+}$ is given. We denote by $\mathbf{u}^{h}$ the corresponding homogeneous deformation

$$\mathbf{u}^{h}(\mathbf{x}) \equiv \mathbf{A}\mathbf{x}.$$

Now let $\Omega = B$, the unit ball in $\mathbb{R}^{n}$, and decompose $B \approx \bigcup_{i=1}^{\infty} \overline{B}_{\varepsilon_i}(\mathbf{x}_i)$ up to (Lebesgue) measure zero as the countable disjoint union of closed balls i.e.

$$\text{meas}\left\{B \backslash \bigcup_{i=1}^{\infty} \overline{B}_{\varepsilon_i}(\mathbf{x}_i)\right\} = 0, \quad \overline{B}_{\varepsilon_i}(\mathbf{x}_i) \cap \overline{B}_{\varepsilon_j}(\mathbf{x}_j) = \emptyset \qquad \text{for } i \neq j, \tag{4}$$

where

$$\overline{B}_{\varepsilon_i}(\mathbf{x}_i) = \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_i| \leq \varepsilon_i\} \subset B$$

and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^{n}$. This can always be achieved for example by Vitali's theorem (see e.g. [6, p. 29]). Note that by (4)

$$\omega_n = \omega_n \sum_{i=1}^{\infty} \varepsilon_i^{n},$$

where ω_n is the volume of the unit ball in $\mathbb{R}^{n}$ and hence

$$\sum_{i=1}^{\infty} \varepsilon_i^{n} = 1. \tag{5}$$

Next define $\tilde{\mathbf{u}} : B \to \mathbb{R}^n$ by

$$\tilde{\mathbf{u}}(\mathbf{x}) = \begin{cases} \varepsilon_i \mathbf{u}\left(\frac{\mathbf{x}-\mathbf{x}_i}{\varepsilon_i}\right) + \mathbf{A}\mathbf{x}_i & \text{if } \mathbf{x} \in B_{\varepsilon_i}(\mathbf{x}_i) \\ \mathbf{A}\mathbf{x} & \text{otherwise.} \end{cases} \tag{6}$$

Then $\tilde{\mathbf{u}} \in W^{1,p}(B;\mathbb{R}^n)$ and $(\tilde{\mathbf{u}} - \mathbf{u}^h) \in W_0^{1,p}(B;\mathbb{R}^n)$ (this construction is used in Proposition 2.3 of Ball & Murat [2]). Moreover, by (6),

$$\begin{aligned} E(\tilde{\mathbf{u}}) &= \int_\Omega W(\nabla\tilde{\mathbf{u}}(\mathbf{x}))\, d\mathbf{x} = \sum_{i=1}^{\infty} \int_{B_{\varepsilon_i}(\mathbf{x}_i)} W\left(\nabla\mathbf{u}\left(\frac{\mathbf{x}-\mathbf{x}_i}{\varepsilon_i}\right)\right) d\mathbf{x} \\ &= \sum_{i=1}^{\infty} \varepsilon_i^n \int_B W(\nabla\mathbf{u}(\mathbf{y}))\, d\mathbf{y} = \int_B W(\nabla\mathbf{u}(\mathbf{y}))\, d\mathbf{y} \end{aligned}$$

(where we have made the change of variables $\mathbf{y} = \frac{\mathbf{x}-\mathbf{x}_i}{\varepsilon_i}$ and used (5)). Therefore $\tilde{\mathbf{u}}$ has the same energy as $\mathbf{u}$. We show in this paper that if $\mathbf{u}$ is a radial cavitation solution then $\tilde{\mathbf{u}}$ is also a *weak solution* of the equations (3) and (2) of nonlinear elasticity. We first gather in the next section some basic results on the radial cavitation problem.

Hypotheses on W

We assume throughout this paper that $W \in C^2(\mathrm{M}_+^{n\times n})$ is isotropic and frame indifferent so that for any $\mathbf{Q} \in \mathrm{SO}(n)$ (the $n \times n$ special orthogonal matrices) we have

$$W(\mathbf{FQ}) = W(\mathbf{QF}) = W(\mathbf{F}) \quad \text{for all } \mathbf{F} \in \mathrm{M}_+^{n\times n}.$$

It is a consequence of the above assumption that

$$W(\mathbf{F}) = \Phi(v_1, v_2, ..., v_n) \quad \text{for all } \mathbf{F} \in \mathrm{M}_+^{n\times n}\ , \tag{7}$$

where Φ is a symmetric function of its arguments and $v_1, v_2, ..., v_n$ denote the singular values of $\mathbf{F}$ (i.e. the eigenvalues of $\sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}}$).

We assume that W is *strongly elliptic*:

$$\frac{\partial^2 W}{\partial F_\alpha^i \partial F_\beta^j}(\mathbf{F}) a^i a^j b^\alpha b^\beta > 0 \tag{8}$$

for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$. We finally assume that there exist $\varepsilon > 0$ and $C > 0$ such that

$$\left\| \mathbf{F}^{\mathrm{T}} \frac{\partial W}{\partial \mathbf{F}}(\mathbf{AF}) \right\| \leq C[W(\mathbf{F}) + 1] \qquad \text{for all } \mathbf{F} \in \mathrm{M}^{n \times n}_{+} \tag{9}$$

whenever $\|\mathbf{A} - \mathbf{I}\| < \varepsilon$. (Here $\| \cdot \|$ denotes the Euclidean norm on $\mathrm{M}^{n \times n}$: $\|\mathbf{F}\|^2 = \mathbf{F} : \mathbf{F}$ and $\mathbf{F} : \mathbf{G} = \mathrm{trace}(\mathbf{F}^{\mathrm{T}}\mathbf{G})$ for $\mathbf{F}, \mathbf{G} \in \mathrm{M}^{n \times n}$.)

Results on Radial Cavitation

Suppose that $\mathbf{A} = \lambda \mathbf{I}$, $\lambda > 0$, $p \in [1, n)$, and

$$\mathbf{u}(\mathbf{x}) = \frac{r(R)}{R}\, \mathbf{x}, \quad R = |\mathbf{x}|, \quad r : [0, 1] \to [0, \infty) \tag{10}$$

is a radial map of B, where $r \in C^2((0, 1]) \cap C^1([0, 1])$. Then

$$\nabla \mathbf{u} = r'(R)\, \frac{\mathbf{x} \otimes \mathbf{x}}{R^2} + \frac{r(R)}{R} \left[\mathbf{I} - \frac{\mathbf{x} \otimes \mathbf{x}}{R^2} \right]$$

and consequently (see e.g. [1]) the singular values, $v_1, v_2, ..., v_n$, of $\mathbf{F} = \nabla \mathbf{u}$ are given by $v_1 = r'(R)$, $v_2 = ... = v_n = \frac{r(R)}{R}$. Thus, using (7),

$$\frac{\partial W}{\partial \mathbf{F}}(\nabla \mathbf{u}) = \Phi_{,1}(R) \frac{\mathbf{x} \otimes \mathbf{x}}{R^2} + \Phi_{,2}(R) \left[\mathbf{I} - \frac{\mathbf{x} \otimes \mathbf{x}}{R^2} \right] \tag{11}$$

and

$$\nabla \mathbf{u}^{\mathrm{T}} \frac{\partial W}{\partial \mathbf{F}}(\nabla \mathbf{u}) = r'(R) \Phi_{,1}(R) \frac{\mathbf{x} \otimes \mathbf{x}}{R^2} + \frac{r(R)}{R} \Phi_{,2}(R) \left[\mathbf{I} - \frac{\mathbf{x} \otimes \mathbf{x}}{R^2} \right], \tag{12}$$

where $\Phi_{,i}$ denotes differentiation of Φ with respect to its i-th argument, and $\Phi(R)$ and $\Phi_{,i}(R)$ denote Φ and $\Phi_{,i}$ evaluated at the arguments $v_1 = r'(R)$, $v_2 = ... = v_n = \frac{r(R)}{R}$.

Now suppose that $\mathbf{u}$, given by (10), *is a radial cavitation map*: i.e. r satisfies

(i) $r(1) = \lambda$,

(ii) $r'(R) > 0 \quad$ for $R \in (0, 1]$,

(iii) the radial equilibrium equation

$$\frac{d}{dR}\left[R^{n-1} \Phi_{,1}(R) \right] = (n-1) R^{n-2} \Phi_{,2}(R) \quad \text{for} \quad R \in (0, 1) \tag{13}$$

(iv) $r(0) > 0$ and $\lim_{R \to 0^+} R^{n-1} \Phi_{,1}(R) = 0$.

Remarks. 1. See [7] and the references therein for constitutive hypotheses on W under which there exist maps corresponding to radial cavitation.

2. A straightforward calculation, using (11) and (12), shows that any solution of the radial equilibrium equation satisfies (2) and (3) in $B\backslash\{\mathbf{0}\}$.

3. The conditions $r'(R) > 0$ for $R \in (0,1]$, $r(0) > 0$, and $r(1) = \lambda > 0$ imply that a map $\mathbf{u}$ that corresponds to radial cavitation must be one-to-one and satisfy $\mathbf{u}(B) \subset \lambda B$.

4. The assumption (8) of strong ellipticity yields $\Phi,_{11} > 0$ for the corresponding function Φ (given by (7)). Thus the radial equilibrium equation is a well-defined second-order differential equation.

5. By [1, Lemma 4.1] a radial map $\mathbf{u}$ of the form (10) lies in $W^{1,p}(B;\mathbb{R}^n)$, $p \geq 1$, if and only if the corresponding scalar function $r(\cdot)$ is absolutely continuous on closed subintervals of $(0,1)$ and satisfies

$$\int_0^1 R^{n-1}\left[|r'(R)|^p + \left|\frac{r(R)}{R}\right|^p\right] dR < \infty \; .$$

The next result notes, in particular, that every radial cavitation map $\mathbf{u}$ has finite energy.

Lemma 1. *Let* $\mathbf{u}$ *be a radial cavitation map (i.e.* $\mathbf{u}$ *is given by (10) where* r *satisfies (13)(i)-(iv)). Then*

$$\begin{aligned}
(i) &\qquad E(\mathbf{u}) = \omega_n\left[\Phi(1) + (r(1) - r'(1))\Phi,_1(1)\right], \\
(ii) &\qquad \delta^n\left[\Phi(\delta) + \left(\frac{r(\delta)}{\delta} - r'(\delta)\right)\Phi,_1(\delta)\right] \to 0 \quad \text{as} \quad \delta \to 0^+, \\
(iii) &\qquad \delta^n\left[\Phi(\delta) - r'(\delta)\Phi,_1(\delta)\right] \to 0 \quad \text{as} \quad \delta \to 0^+,
\end{aligned}$$

Proof. Parts (i) and (ii) of the lemma (stated for $n \geq 2$) follow from a straightforward generalisation of Proposition 1.13 in [10] (which proves the same results, under hypothesis (8), in the case $n = 3$). Part (iii) is then an immediate consequence of part (ii) and (13)(iv). □

Lemma 2. *The radial cavitation map* $\mathbf{u}$ *satisfies the weak form of the energy momentum equations:*

$$\int_B \nabla\mathbf{v} : \left[W(\nabla\mathbf{u})\mathbf{I} - \nabla\mathbf{u}^{\mathrm{T}}\frac{\partial W}{\partial \mathbf{F}}(\nabla\mathbf{u})\right] d\mathbf{x} = 0 \qquad \textit{for all } \mathbf{v} \in C_0^1(B;\mathbb{R}^n).$$

Remark. The above equations arise as necessary conditions for a minimiser of (1) on using inner variations and the condition (9) (see [3]).

Proof. We first note that by (9), the fact that $\mathbf{u}$ has finite energy, and the dominated convergence theorem

$$\int_B \nabla\mathbf{v} : \mathbf{M}(\nabla\mathbf{u})\, d\mathbf{x} = \lim_{\delta\to 0^+} \int_{B\setminus B_\delta(\mathbf{0})} \nabla\mathbf{v} :\ \mathbf{M}(\nabla\mathbf{u})\, d\mathbf{x}, \tag{14}$$

where

$$\mathbf{M}(\nabla\mathbf{u}) = W(\nabla\mathbf{u})\mathbf{I} - \nabla\mathbf{u}^{\mathrm{T}}\frac{\partial W}{\partial\mathbf{F}}(\nabla\mathbf{u}). \tag{15}$$

Next, since $\mathbf{u}$ satisfies (2) in $B\setminus\{\mathbf{0}\}$, it follows from (12) and the divergence theorem that

$$\begin{aligned}
\int_{B\setminus B_\delta(\mathbf{0})} \nabla\mathbf{v} :\ \mathbf{M}(\nabla\mathbf{u})\, d\mathbf{x} &= \int_{\partial B\cup\partial B_\delta(\mathbf{0})} \mathbf{v}\cdot\mathbf{M}(\nabla\mathbf{u})\mathbf{n} \\
&= \int_{\partial B_\delta(\mathbf{0})} -\mathbf{v}\cdot\mathbf{M}(\nabla\mathbf{u})\frac{\mathbf{x}}{|\mathbf{x}|} \\
&= \int_{\partial B_\delta(\mathbf{0})} \mathbf{v}\cdot\left[\Phi(R) - r'(R)\Phi_{,1}(R)\right]\mathbf{n} \\
&= \left[\Phi(\delta) - r'(\delta)\Phi_{,1}(\delta)\right]\int_{B_\delta(\mathbf{0})} \operatorname{div}\mathbf{v}\, d\mathbf{x}.
\end{aligned}$$

However,

$$\left|\left|\Phi(\delta) - r'(\delta)\Phi_1(\delta)\right|\int_{B_\delta(\mathbf{0})} \operatorname{div}\mathbf{v}\, d\mathbf{x}\right| \le C\delta^n\left|\Phi(\delta) - r'(\delta)\Phi_{,1}(\delta)\right| \to 0$$

as $\delta\to 0^+$ by Lemma 1, which together with (14) and the above calculation yields the desired result. $\square$

A slight modification of the above argument yields the following result which holds in the case when the test function $\mathbf{v}$ is not necessarily zero on ∂B.

Lemma 3. *The radial cavitation map* $\mathbf{u}$ *satisfies*

$$\int_B \nabla\mathbf{v} : \left[W(\nabla\mathbf{u})\mathbf{I} - \nabla\mathbf{u}^{\mathrm{T}}\frac{\partial W}{\partial\mathbf{F}}(\nabla\mathbf{u})\right] d\mathbf{x} = \left[\Phi(1) - r'(1)\Phi_{,1}(1)\right]\int_{\partial B} \mathbf{v}\cdot\mathbf{n}$$

for all $\mathbf{v}\in C^1(\overline{B};\mathbb{R}^n)$.

Remark. It is a consequence of the invariance of the equilibrium equations under the scaling symmetry $(\mathbf{x}, \mathbf{u}) \to (\epsilon\mathbf{x}, \epsilon\mathbf{u})$ $(\epsilon > 0)$, that any weak solution $\mathbf{u} : B \to \mathbb{R}^n$ of (2) (or (3)) generates a corresponding weak solution $\mathbf{u}_\epsilon(\mathbf{x}) = \epsilon\mathbf{u}\left(\frac{\mathbf{x}}{\epsilon}\right)$ of (2) (or (3)) on the domain ϵB.

We note further for later use that in the particular case that $\mathbf{u}$ is a radial cavitation map we also have the following consequence of Lemma 3

$$\int_{\epsilon B} \nabla\mathbf{v} : \left[W(\nabla\mathbf{u}_\epsilon)\mathbf{I} - \nabla\mathbf{u}_\epsilon^{\mathrm{T}}\frac{\partial W}{\partial \mathbf{F}}(\nabla\mathbf{u}_\epsilon)\right] d\mathbf{x} = \left[\Phi(1) - r'(1)\Phi_{,1}(1)\right] \int_{\partial(\epsilon B)} \mathbf{v}\cdot\mathbf{n}$$

for all $\mathbf{v} \in C^1(\epsilon\overline{B}; \mathbb{R}^n)$.

The next two results correspond to Lemmas 2 and 3 for the second form (3) of the equilibrium equations. These follow by analogous arguments to the above and so the proofs are omitted.

Lemma 4. *The radial cavitation map satisfies the weak form of the equilibrium equations (3) i.e.*

$$\int_B \nabla\mathbf{v} : \frac{\partial W}{\partial \mathbf{F}}(\nabla\mathbf{u})\, d\mathbf{x} = 0 \qquad \textit{for all } \mathbf{v} \in C_0^1(B; \mathbb{R}^n).$$

Lemma 5. *The radial cavitation map satisfies*

$$\int_B \nabla\mathbf{v} : \frac{\partial W}{\partial \mathbf{F}}(\nabla\mathbf{u})\, d\mathbf{x} = \Phi_{,1}(1) \int_{\partial B} \mathbf{v}\cdot\mathbf{n} \qquad \textit{for all } \mathbf{v} \in C^1(\overline{B}; \mathbb{R}^n).$$

Remark. Lemmas 4 and 5 can also be obtained from Theorem 4.2 in [1].

Construction of infinitely many singular weak solutions

We next apply the construction outlined earlier in (4)–(6) to a radial cavitation map $\mathbf{u}$ to construct a corresponding map $\tilde{\mathbf{u}} : B \to \mathbb{R}^n$ with infinitely many discontinuities. The main results of this paper are the following two theorems.

Theorem 1. *The map $\tilde{\mathbf{u}}$ lies in $W^{1,p}(B; \mathbb{R}^n)$ for $1 \le p < n$, is injective almost everywhere, and is a weak solution of the equations*

$$\frac{\partial}{\partial x^\alpha}\left[W(\nabla\mathbf{u})\,\delta_\alpha^\beta - u_{,\beta}^{k}\,\frac{\partial W}{\partial F_\alpha^k}(\nabla\mathbf{u})\right] = 0 \quad \text{in } B, \quad \beta = 1, 2, \ldots, n$$

i.e.

$$\int_B \nabla \mathbf{v} : \left[W(\nabla \tilde{\mathbf{u}})\mathbf{I} - \nabla \tilde{\mathbf{u}}^{\mathrm{T}} \frac{\partial W}{\partial \mathbf{F}}(\nabla \tilde{\mathbf{u}}) \right] d\mathbf{x} = 0 \qquad \textit{for all } \mathbf{v} \in C_0^1(B; \mathbb{R}^n).$$

Proof. The fact that $\tilde{\mathbf{u}} \in W^{1,p}(B; \mathbb{R}^n)$ and is injective almost everywhere follows directly (see Remarks 3 and 5 which precede Lemma 1 in this paper). Now let $\mathbf{v} \in C_0^1(B; \mathbb{R}^n)$. Then by the definition of $\tilde{\mathbf{u}}$, (15), Lemma 3 (and the remark following)

$$\begin{aligned}
\int_B \nabla \mathbf{v} : \mathbf{M}(\nabla \tilde{\mathbf{u}})\, d\mathbf{x} &= \sum_{i=1}^{\infty} \int_{B_{\varepsilon_i}(\mathbf{x}_i)} \nabla \mathbf{v} : \mathbf{M}(\nabla \tilde{\mathbf{u}})\, d\mathbf{x} \\
&= \sum_{i=1}^{\infty} \left[\Phi(1) - r'(1)\Phi_{,1}(1) \right] \int_{\partial B_{\varepsilon_i}(\mathbf{x}_i)} \mathbf{v} \cdot \mathbf{n} \\
&= \left[\Phi(1) - r'(1)\Phi_{,1}(1) \right] \sum_{i=1}^{\infty} \int_{B_{\varepsilon_i}(\mathbf{x}_i)} \operatorname{div} \mathbf{v}\, d\mathbf{x} \\
&= \left[\Phi(1) - r'(1)\Phi_{,1}(1) \right] \int_B \operatorname{div} \mathbf{v}\, d\mathbf{x} \\
&= \left[\Phi(1) - r'(1)\Phi_{,1}(1) \right] \int_{\partial B} \mathbf{v} \cdot \mathbf{n} \;=\; 0.
\end{aligned}$$

□

Theorem 2. *The map* $\tilde{\mathbf{u}}$ *is a weak solution of the equations*

$$\frac{\partial}{\partial x^\alpha} \left[\frac{\partial W}{\partial F^i_\alpha}(\nabla \mathbf{u}) \right] = 0 \quad \text{in } B, \quad i = 1, 2, \ldots, n$$

i.e.

$$\int_B \nabla \mathbf{v} : \frac{\partial W}{\partial \mathbf{F}}(\nabla \tilde{\mathbf{u}})\, d\mathbf{x} = 0 \qquad \textit{for all } \mathbf{v} \in C_0^1(B; \mathbb{R}^n).$$

Proof. The proof of this result is exactly analogous to that of the last theorem and is therefore omitted. □

Concluding Remarks

We note that the construction used in the last two theorems is independent of the domain chosen and therefore yields singular weak solutions for any domain (given a domain Ω write $\Omega \approx \bigcup_{i=1}^{\infty} \overline{B}_{\varepsilon_i}(\mathbf{x}_i)$ up to measure zero, define $\tilde{\mathbf{u}}$ by (6), and proceed as in the case $\Omega = B$).

Note that, in the case when $\mathbf{u}$ is a radial cavitation map, although the "composite deformation" $\tilde{\mathbf{u}}$ constructed according to (6) is continuous away from the singular set $\{\mathbf{x}_i : i \in \mathbb{N}\}$ it is not C^1 (see the next paragraph). This has implications for proving regularity of weak solutions in nonlinear elasticity (see e.g. Evans [5]). In this context we mention the recent interesting example of Müller & Šverák [9] of a $W^{1,\infty}$ weak solution which is nowhere C^1.

In particular $\tilde{\mathbf{u}}$ is not C^1 on the boundary of the balls $B_{\epsilon_k}(\mathbf{x}_k)$. This follows from $(6)_1$ since $\|\nabla \mathbf{u}\|$ is infinite at the origin and each $\mathbf{x}^* \in \partial B_{\epsilon_k}(\mathbf{x}_k)$ is an accumulation point of the centres of the balls. (Otherwise, if $\mathbf{x}^* \in \partial \overline{B}_{\varepsilon_1}(\mathbf{x}_1)$ with $\mathbf{x}^* \notin \overline{\{\mathbf{x}_i\}}$ then there is a $\delta > 0$ such that $\mathbf{x}_i \notin B_{3\delta}(\overline{\mathbf{x}})$ for all i. By (5) there exists $M \in \mathbb{N}$ such that $\varepsilon_i < \delta$ and hence $B_\delta(\mathbf{x}^*) \cap \overline{B}_{\varepsilon_i}(\mathbf{x}_i) = \emptyset$ for $i > M$. Also, $\mathbf{x}^* \notin K := \bigcup_{i=2}^{M} \overline{B}_{\varepsilon_i}(\mathbf{x}_i)$, which is closed, so $d := \operatorname{dist}(\mathbf{x}^*, K) > 0$. Consequently, $D := B_{\min(d,\delta)}(\mathbf{x}^*) \backslash \overline{B}_{\varepsilon_1}(\mathbf{x}_1)$ has positive measure and $D \cap \overline{B}_{\varepsilon_i}(\mathbf{x}_i) = \emptyset$ for every i, a contradiction.)

Though it is well known to workers in nonlinear elasticity that the construction (6) yields deformations with equal energy it seems to have been (surprisingly) overlooked to date that in the case when $\mathbf{u}$ is a radial cavitation map this construction also produces further weak solutions.

Finally, we note that an analogous approach to that used in this paper can be used to show that the maps $\tilde{\mathbf{u}}$ constructed from rescalings of a radial cavitation map generate weak solutions of the Cauchy form of the equilibrium equations:

$$\frac{\partial}{\partial y^\alpha} T_{i\alpha}(\mathbf{y}) = 0, \qquad \mathbf{y} \in \mathbf{u}(\Omega),$$

where

$$T_{i\alpha}(\mathbf{u}(\mathbf{x})) = \frac{1}{\det \nabla \mathbf{u}(\mathbf{x})} \frac{\partial W}{\partial F^i_\beta}(\nabla \mathbf{u}(\mathbf{x})) u,^\alpha_\beta(\mathbf{x}).$$

The details are left to the interested reader. (Though of course in this case the deformed body $\tilde{\mathbf{u}}(\Omega)$ will vary according to the particular decomposition of Ω used in constructing $\tilde{\mathbf{u}}$.)

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