Energy Minimization for an Elastic Fluid

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Abstract

For a homogeneous elastic fluid, the specific internal energy function depends on the deformation gradient through the specific volume, which is essentially the determinant of the deformation gradient. Standard, elementary minimization problems are commonly stated in terms of the specific volume field and either the body of fluid is subject to a constant environmental pressure or the total volume is fixed. When the deformation field, itself, is of primary concern, the specific internal energy function is left expressed as a function of the deformation gradient and the standard basic minimization problem in elasticity is to minimize the total internal energy of the body subject to a given homogeneous deformation on its boundary. We discuss the relationship between these two basic problems. Finally, we provide a solution to the latter minimization problem when the specified boundary placement data is arbitrary but sufficiently regular. Our analysis excludes the possibility of cavitation.

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1 Introduction

The static deformation of a body is represented by a mapping of points $\mathbf{x} \in \mathcal{B}_0 \subset \mathbb{E}^3$ to points $\mathbf{y} = \mathbf{y}(\mathbf{x}) \in \mathcal{B} \subset \mathbb{E}^3$. We use $\mathcal{B}_0$ to denote a natural reference configuration of the body (undistorted and stress-free) and $\mathcal{B} = \mathbf{y}(\mathcal{B}_0)$ to denote its (relatively) distorted configuration. The mapping is supposed to be oriented and it is assumed that the deformation gradient $\mathbf{F} = \mathbf{F}(\mathbf{x}) := \nabla \mathbf{y}(\mathbf{x})$ satisfies $\det \mathbf{F}(\mathbf{x}) > 0$: We denote this as $\mathbf{F} \in \mathrm{Lin}^+$, where Lin represents the set of all linear transformation of $\mathbb{E}^3 \to \mathbb{E}^3$. For a homogeneous, elastic body, the constant referential mass density of $\mathcal{B}_0$, i.e., $\rho_0 > 0$, and the mass density of $\mathcal{B}$, i.e., $\rho = \rho(\mathbf{x})$, satisfy the balance of mass $\rho = \rho_0 / \det \mathbf{F}$. In addition, the specific internal energy field $\varepsilon = \varepsilon(\mathbf{x})$, measured per unit mass of the body, is given by a constitutive assumption of the form $\varepsilon = \hat{\varepsilon}(\mathbf{F})$, where $\hat{\varepsilon}(\cdot)$ is defined for all $\mathbf{F} \in \mathrm{Lin}^+$. If the body is an **elastic fluid** then we may write

$$\varepsilon = \hat{\varepsilon}(\mathbf{F}) = \bar{\varepsilon}(\nu), \tag{1.1}$$

where $\nu = 1/\rho = \nu_0 \det \mathbf{F}$ is the specific volume of $\mathcal{B}$, with $\nu_0 = 1/\rho_0$ being the specific volume of the reference configuration $\mathcal{B}_0$. Thus, by the chain rule we have

$$D_{\mathbf{F}} \hat{\varepsilon}(\mathbf{F}) = \nu D_\nu \bar{\varepsilon}(\nu) \mathbf{F}^{-\mathrm{T}}, \tag{1.2}$$

which implies that the Cauchy stress is given by

$$\mathbf{T} := \rho D_{\mathbf{F}} \hat{\varepsilon}(\mathbf{F}) \mathbf{F}^{\mathrm{T}} = D_\nu \bar{\varepsilon}(\nu) \mathbf{1}. \tag{1.3}$$

In this work, we are interested mainly in the relationship between the following two minimization problems for an elastic fluid body. In the first problem, the fundamental field to determine is the deformation of the body and the boundary placement is prescribed*; in the second problem, the fundamental field to determine is the specific volume and the total volume of the distorted body is prescribed. It seems natural to believe that the two problems are equivalent if, in both cases, the total volumes of the distorted body are equal, and we investigate this notion of "equivalence".

Problem P1: Given a homogeneous, elastic body and a natural reference configuration $\mathcal{B}_0$ with constant mass density $\rho_0 > 0$. Minimize the total stored energy of the body subject to a specified homogeneous deformation at its boundary, i.e.,

$$\underset{\mathbf{y}(\cdot) \in \mathcal{A}_1}{\mathsf{minimize}} \int_{\mathcal{B}_0} \rho_0 \hat{\varepsilon}(\nabla \mathbf{y}(\mathbf{x}))\, dv, \quad \text{with } \mathbf{y}(\mathbf{x}) = \mathbf{F}^* \mathbf{x} \quad \forall\, \mathbf{x} \in \partial \mathcal{B}_0, \tag{1.4a}$$

where $\mathbf{F}^* \in \mathrm{Lin}^+$ is given and where

*Here, the boundary placement is restricted to be given by a linear transformation of the referential boundary points. However, in Section 4, we significantly generalize the form that the prescribed boundary placement may have.

$$\mathcal{A}_1 := \{\mathbf{y}(\cdot) \in W^{1,p}(\mathcal{B}_0) |\ \mathbf{y}(\mathbf{x}) \text{ is 1-to-1 a.e., } \det \nabla\mathbf{y}(\mathbf{x}) > 0 \text{ a.e., } \mathbf{x} \in \mathcal{B}_0\}, \tag{1.4b}$$

with $p \geq 3$.[†]

Problem P2: Given a homogeneous, elastic fluid body and a natural reference configuration $\mathcal{B}_0$ with constant mass density $\rho_0 > 0$. Minimize the total stored energy of the body subject to a specified total volume, i.e.,

$$\underset{\nu(\cdot)\in\mathcal{A}_2}{\text{minimize}} \int_{\mathcal{B}_0} \rho_0 \overline{\varepsilon}(\nu(\mathbf{x}))\, dv, \quad \text{with} \quad \int_{\mathcal{B}_0} \rho_0 \nu(\mathbf{x})\, dv = \operatorname{vol}\mathcal{B}, \tag{1.5a}$$

where $\operatorname{vol}\mathcal{B}$ denotes the prescribed volume of the distorted configuration $\mathcal{B}$ and

$$\mathcal{A}_2 := \{\nu(\cdot) \in L^1(\mathcal{B}_0) |\ \nu(\mathbf{x}) > 0 \text{ a.e., } \mathbf{x} \in \mathcal{B}_0\}. \tag{1.5b}$$

In order to connect these two problems together, first let $\mathcal{B} = \{\mathbf{F}^*\mathbf{x} \mid \mathbf{x} \in \mathcal{B}_0\}$ and note that, for a sufficiently smooth admissible mapping $\mathbf{y} = \mathbf{y}(\mathbf{X})$ involved in Problem P1,

$$\begin{aligned}
\operatorname{vol}\mathcal{B} = \int_{\mathcal{B}_0} \det \nabla\mathbf{y}(\mathbf{x})\, dv &= \frac{1}{3}\int_{\mathcal{B}_0} \operatorname{div}\big((\operatorname{adj}\nabla\mathbf{y})\mathbf{y}\big)\, dv \\
&= \frac{1}{3}\int_{\partial\mathcal{B}_0} \mathbf{y}\cdot\big((\operatorname{adj}\nabla\mathbf{y})^{\mathrm{T}}\mathbf{n}\big)\, da \\
&= \frac{1}{3}\int_{\partial\mathcal{B}_0} \mathbf{F}^*\mathbf{x}\cdot\operatorname{cof}\mathbf{F}^*\mathbf{n}\, da \\
&= \frac{1}{3}\det\mathbf{F}^*\int_{\partial\mathcal{B}_0} \mathbf{x}\cdot\mathbf{n}\, da \\
&= \det\mathbf{F}^*\operatorname{vol}\mathcal{B}_0 = \int_{\mathcal{B}_0} \det\mathbf{F}^*\, dv,
\end{aligned} \tag{1.6}$$

where $\mathbf{n}$ denotes the outer unit normals on $\partial\mathcal{B}_0$ and $\operatorname{adj}\mathbf{F} = (\operatorname{cof}\mathbf{F})^{\mathrm{T}}$ is the adjugate of $\mathbf{F}$ (i.e., the unique element in Lin^+ satisfying $\mathbf{F}(\operatorname{adj}\mathbf{F}) = (\det\mathbf{F})\mathbf{1})$[‡]. Accordingly, for the corresponding Problem P2, an admissible specific volume field $\nu(\mathbf{x})$ must satisfy the constraint

[†] The restriction $p \geq 3$ is a technical requirement, which in particular guarantees that $\det\nabla\mathbf{u}(\cdot) \in L^1(\mathcal{B}_0)$, $\forall\, \mathbf{u}(\cdot) \in \mathcal{A}_1$ and eliminates the possibility of cavitation for Problem P1, a phenomenon that is not of major relevance in this current work.

[‡] The result for a general Sobolev map in $\mathcal{A}_1$ follows by approximation.

$$\int_{\mathcal{B}_0} \rho_0 \nu(\mathbf{x})\, dv = \nu^* \rho_0 \mathrm{vol}\, \mathcal{B}_0 \quad \text{or} \quad \int_{\mathcal{B}_0} \rho_0 \big(\nu(\mathbf{x}) - \nu^*\big)\, dv = 0, \tag{1.7}$$

where $\nu^* := \nu_0 \det \mathbf{F}^*$ denotes a prescribed specific volume corresponding to the given $\mathbf{F}^*$ for Problem P1. Note that the expression $\rho_0 \mathrm{vol}\, \mathcal{B}_0$ in (1.7) corresponds to the total fixed mass of the body.

In Section 2, we record a few elementary convexity and rank 1 convexity issues that are characteristic for an elastic fluid and that are relevant to this work.

In Section 3, we develop an explicit minimizer for Problem P1 for a body of spherical shape and then we use a scaling and covering strategy to show how this minimizer generates a minimizer for an arbitrarily shaped body. We characterize the complete class of boundary placements for which a minimizer of Problem P1 is composed of an arbitrarily fine "mixture" of two kinds of deformation gradients. This reflects the type of two-phase minimizing states that are well-known solutions to Problem P2 for a certain range of specified distorted volumes.[§]

In Section 4, we introduce a Generalized (minimization) Problem GP1 which replaces the specified boundary placement of Problem P1 by an 'arbitrarily' specified form. We then show how a theorem of Dacorogna and Moser [7], together with a strategy of scaling and covering, is used to generate a solution to this more general problem for the arbitrarily shaped domain.[¶] We close by showing that the minimizers of this work satisfy weak forms of the equation of equilibrium and the energy-momentum equation.

2 Convexity issues

The function $\hat{\varepsilon}(\cdot)$ is said to be **rank 1 convex at** $\tilde{\mathbf{F}} \in \mathrm{Lin}^+$ if

$$\hat{\varepsilon}(\mathbf{F}) \geq \hat{\varepsilon}(\tilde{\mathbf{F}}) + D_{\mathbf{F}}\hat{\varepsilon}(\tilde{\mathbf{F}}) \cdot (\mathbf{F} - \tilde{\mathbf{F}}) \tag{2.1}$$

for all $\mathbf{F} \in \mathrm{Lin}^+$ such that $\mathbf{F} - \tilde{\mathbf{F}} = \mathbf{a} \otimes \mathbf{b}$, where $\mathbf{b}$ is any unit vector and $\mathbf{a}$ is any vector, both in $\mathbb{E}^3$. The function $\overline{\varepsilon}(\cdot)$ is said to be **convex at** $\tilde{\nu} \in \mathbb{R}^+$ if

[§]Mizel [9] also considers a version of Problem P1 for an isotropic elastic solid whose specific internal energy $\hat{\varepsilon}(\cdot)$ is rank 1 convex, but $A(\delta) := \hat{\varepsilon}(\delta^3 \mathbf{1})$, for $\delta \in (0, \infty)$, is not convex. He relates his minimizers to the phenomenon of fracture.

[¶]For interesting general analytical results related to the existence of minimizers for the Generalized Problem GP1 and the corresponding "relaxed" problem, we refer to Dacorogna [6], Cupini and Mascolo [5], Cellini and Zagatti [4] and the references therein. While our motivation in Section 4 is somewhat more physical, our argument in support of our final Theorem 4.1 follows a path similar to Dacorogna [6], Corollary 14.9, but requires less smoothness for the boundary $\partial\mathcal{B}_0$ of $\mathcal{B}_0$ than is stated in this corollary.

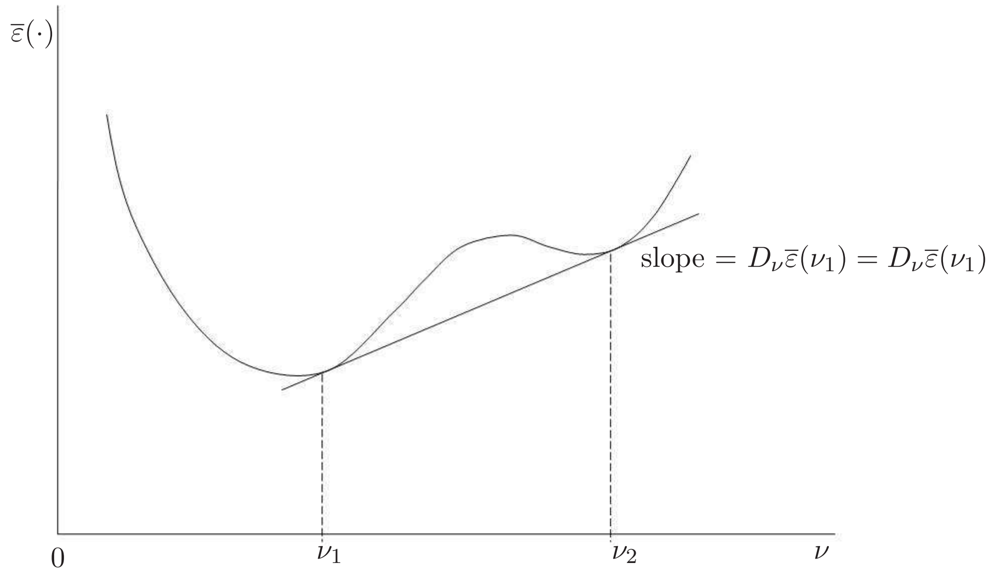

Figure 1: Graph of $\overline{\varepsilon}(\cdot)$.

$$\overline{\varepsilon}(\nu) \geq \overline{\varepsilon}(\tilde{\nu}) + D_\nu\overline{\varepsilon}(\tilde{\nu})(\nu - \tilde{\nu}) \tag{2.2}$$

for all $\nu \in \mathbb{R}^+$. However, with the identification $\nu = \nu_0 \det \mathbf{F}$ and $\tilde{\nu} = \nu_0 \det \tilde{\mathbf{F}}$, it is easy to see that for an elastic fluid, for which (1.1) holds, we have

Theorem 2.1 *The function $\hat{\varepsilon}(\cdot)$ is rank 1 convex at $\tilde{\mathbf{F}} \in \mathrm{Lin}^+$ if and only if the function $\overline{\varepsilon}(\cdot)$ is convex at $\tilde{\nu} \in \mathbb{R}^+$.*

Proof: First, write $\mathbf{F} = \tilde{\mathbf{F}} + \mathbf{a}\otimes\mathbf{b} = \tilde{\mathbf{F}}(\mathbf{1} + \tilde{\mathbf{F}}^{-1}\mathbf{a}\otimes\mathbf{b})$ and take the determinant to reach $\nu = \tilde{\nu}(1 + \tilde{\mathbf{F}}^{-1}\mathbf{a}\cdot\mathbf{b})$. Then, note that $\tilde{\nu}\tilde{\mathbf{F}}^{-\mathrm{T}}\cdot(\mathbf{F} - \tilde{\mathbf{F}}) = \tilde{\nu}\tilde{\mathbf{F}}^{-\mathrm{T}}\cdot(\mathbf{a}\otimes\mathbf{b}) = \tilde{\nu}\tilde{\mathbf{F}}^{-1}\mathbf{a}\cdot\mathbf{b} = \nu - \tilde{\nu}$, and observe, using (1.2), that

$$\hat{\varepsilon}(\mathbf{F}) - \Big(\hat{\varepsilon}(\tilde{\mathbf{F}}) + D_{\mathbf{F}}\hat{\varepsilon}(\tilde{\mathbf{F}})\cdot(\mathbf{F} - \tilde{\mathbf{F}})\Big) = \overline{\varepsilon}(\nu) - \Big(\overline{\varepsilon}(\tilde{\nu}) + D_\nu\overline{\varepsilon}(\tilde{\nu})(\nu - \tilde{\nu})\Big), \tag{2.3}$$

to complete the proof. ■

Throughout this paper we assume that $\hat{\varepsilon}(\cdot)$ and $\overline{\varepsilon}(\cdot)$ are smooth functions and that $\overline{\varepsilon}(\cdot)$ is convex at all points $\tilde{\nu} \in \mathbb{R}^+$ outside the open interval (ν_1, ν_2), as shown in Figure 1. Thus, it follows that

$$D_\nu \bar{\varepsilon}(\nu_1) = D_\nu \bar{\varepsilon}(\nu_2) \tag{2.4a}$$

and that

$$\bar{\varepsilon}(\nu_1) - D_\nu \bar{\varepsilon}(\nu_1)\nu_1 = \bar{\varepsilon}(\nu_2) - D_\nu \bar{\varepsilon}(\nu_2)\nu_2. \tag{2.4b}$$

Let us define the surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ in Lin^+ through

$$\mathcal{S}_1 := \{\mathbf{F} \in \mathrm{Lin}^+ |\ \nu_0 \det \mathbf{F} = \nu_1\}, \quad \mathcal{S}_2 := \{\mathbf{F} \in \mathrm{Lin}^+ |\ \nu_0 \det \mathbf{F} = \nu_2\}. \tag{2.5}$$

Then, because of Theorem 2.1 it follows that $\hat{\varepsilon}(\cdot)$ is rank 1 convex for all $\tilde{\mathbf{F}} \in \mathrm{Lin}^+$ such that $\nu_0 \det \tilde{\mathbf{F}} \leq \nu_1$ and $\nu_0 \det \tilde{\mathbf{F}} \geq \nu_2$. Moreover, we have

Theorem 2.2 *Suppose $\mathbf{F}_2 \in \mathcal{S}_2$ is given. Then, the points $\mathbf{F}_1 \in \mathcal{S}_1$ that are rank 1 connected to $\mathbf{F}_2$ are completely characterized by*

$$\mathbf{F}_1 = \mathbf{F}_2 \left(\mathbf{1} + \frac{\nu_1 - \nu_2}{\nu_2} \mathbf{b} \otimes \mathbf{b} + \alpha \mathbf{b}^\perp \otimes \mathbf{b} \right), \tag{2.6}$$

where $\alpha \in \mathbb{R}$ is arbitrary, $\mathbf{b}$ is an arbitrary unit vector in $\mathbb{E}^3$ and $\mathbf{b}^\perp$ is orthogonal to $\mathbf{b}$.

Proof: Observe that since we wish $\mathbf{F}_1 - \mathbf{F}_2 = \mathbf{a} \otimes \mathbf{b}$, then $\det \mathbf{F}_1 = \det \mathbf{F}_2 (1 + \mathbf{F}_2^{-1}\mathbf{a} \cdot \mathbf{b})$. Moreover, note that because $\mathbf{F}_1 \in \mathcal{S}_1$ and $\mathbf{F}_2 \in \mathcal{S}_2$ we may then write

$$\mathbf{F}_2^{-1}\mathbf{a} \cdot \mathbf{b} = \frac{\nu_1 - \nu_2}{\nu_2}.$$

Thus, for every unit vector $\mathbf{b}$ we have

$$\mathbf{F}_2^{-1}\mathbf{a} = \frac{\nu_1 - \nu_2}{\nu_2}\mathbf{b} + \alpha \mathbf{b}^\perp,$$

which readily yields the form of $\mathbf{a}$ and completes the proof. ■

Now, let us suppose that $\mathbf{F}^* \in \mathrm{Lin}^+$ is such that $\nu^* := \nu_0 \det \mathbf{F}^* \in (\nu_1, \nu_2)$. Thus, $\mathbf{F}^*$ is a point in Lin^+ where $\hat{\varepsilon}(\cdot)$ is not rank 1 convex. Clearly, we have

$$\nu^* = \mu\nu_1 + (1 - \mu)\nu_2, \quad \text{or,} \quad \mu = \frac{\nu^* - \nu_2}{\nu_1 - \nu_2}. \tag{2.7}$$

It follows that there exists a point $\mathbf{F}_2 \in \mathcal{S}_2$ and a corresponding point $\mathbf{F}_1 \in \mathcal{S}_1$ given by (2.6) such that

$$\mathbf{F}^* = \mu \mathbf{F}_1 + (1-\mu)\mathbf{F}_2. \tag{2.8}$$

In fact, it is easy to show that

$$\mathbf{F}_2 = \mathbf{F}^* \Big(\mathbf{1} - \mu \frac{\nu_2}{\nu *} \big(\frac{\nu_1 - \nu_2}{\nu_2} \mathbf{b} + \alpha \mathbf{b}^\perp \big) \otimes \mathbf{b} \Big) \tag{2.9a}$$

and

$$\mathbf{F}_1 = \mathbf{F}^* \Big(\mathbf{1} + (1-\mu) \frac{\nu_2}{\nu *} \big(\frac{\nu_1 - \nu_2}{\nu_2} \mathbf{b} + \alpha \mathbf{b}^\perp \big) \otimes \mathbf{b} \Big). \tag{2.9b}$$

From a geometric point of view, the set of points in Lin^+ between the two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ are points where the function $\hat{\varepsilon}(\cdot)$ fails to be rank 1 convex.

3 The minimization problems

The minimization Problem P2 is elementary and has been thoroughly discussed in the literature. Thus, we turn our main attention to the minimization Problem P1 and its relationship to Problem P2. We split our discussion into the following two cases:

Case 1: $\mathbf{F}^*$ is a rank 1 convex point of $\hat{\varepsilon}(\cdot)$

In this case, $\nu^* = \nu_0 \det \mathbf{F}^*$ is a convex point of $\overline{\varepsilon}(\cdot)$, and, according to (1.5) and (1.7),

$$\nu^* = \frac{1}{\mathrm{vol}\, \mathcal{B}_0} \int_{\mathcal{B}_0} \nu(\mathbf{x})\, dv. \tag{3.1}$$

Thus, by (2.2) and Theorem 2.1, a minimizer of Problem P2 is $\tilde{\nu}(\mathbf{x}) \equiv \nu^*$.

Now, suppose $\mathbf{y}(\mathbf{x})$ is any deformation field with $\nabla \mathbf{y}(\mathbf{x}) \in \mathrm{Lin}^+$ which satisfies $\mathbf{y}(\mathbf{x}) = \mathbf{F}^*\mathbf{x}$ on $\partial \mathcal{B}_0$, and suppose $\tilde{\mathbf{y}}(\mathbf{x}) := \mathbf{F}^*\mathbf{x}$ for all $\mathbf{x} \in \mathcal{B}_0$. Then, according to (1.1) and (2.2), we may write

$$\int_{\mathcal{B}_0} \rho_0 \Big(\hat{\varepsilon}\big(\nabla \mathbf{y}(\mathbf{x})\big) - \hat{\varepsilon}\big(\mathbf{F}^*\big) \Big)\, dv = \int_{\mathcal{B}_0} \rho_0 \Big(\overline{\varepsilon}(\nu(\mathbf{x})) - \overline{\varepsilon}(\nu^*) \Big)\, dv \geq \int_{\mathcal{B}_0} \rho_0 D_\nu \overline{\varepsilon}(\nu^*) \big(\nu(\mathbf{x}) - \nu^*\big)\, dv.$$

However $D_\nu\bar{\varepsilon}(\nu^*)$ is constant, and because of (1.7) the integral on the right hand side above vanishes. Thus, we see that $\tilde{\mathbf{y}}(\mathbf{x}) := \mathbf{F}^*\mathbf{x}$ is a minimizer of Problem P1 and we observe that $\nu_0 \det \nabla\tilde{\mathbf{y}}(\mathbf{x}) = \nu^* = \tilde{\nu}(\mathbf{x})$.

Case 2: $\mathbf{F}^*$ is not a rank 1 convex point of $\hat{\varepsilon}(\cdot)$

In this case, $\nu^* = \nu_0 \det \mathbf{F}^* \in (\nu_1, \nu_2)$ and $\mathbf{F}^*$ lies in the region of Lin^+ between the surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$. Also, (2.7), (2.8) and (2.9) apply. Moreover, it is well-known that in this case a minimizer of Problem P2 is given by

$$\tilde{\nu}(\mathbf{x}) = \chi_{\mathcal{P}_0}(\mathbf{x})\nu_1 + \big(1 - \chi_{\mathcal{P}_0}(\mathbf{x})\big)\nu_2, \tag{3.2}$$

where $\chi_{\mathcal{P}_0}(\cdot)$ is the characteristic function for the set $\mathcal{P}_0 \subset \mathcal{B}_0$ whose volume is determined by (2.7) and

$$\mu = \frac{\nu^* - \nu_2}{\nu_1 - \nu_2} = \frac{\mathrm{vol}\,\mathcal{P}_0}{\mathrm{vol}\,\mathcal{B}_0}; \tag{3.3}$$

this readily follows from the volume constraint requirement of Problem P2, expressed equivalently in (1.7), i.e.,

$$\int_{\mathcal{B}_0} \rho_0\tilde{\nu}(\mathbf{x})\,dv = \rho_0(\mathrm{vol}\,\mathcal{P}_0)\nu_1 + \rho_0(1 - \mathrm{vol}\,\mathcal{P}_0)\nu_2 = \nu^*\rho_0\mathrm{vol}\,\mathcal{B}_0.$$

Thus, for Problem P1, suppose $\tilde{\mathbf{y}}(\mathbf{x}) \in \mathcal{A}_1$ is a function that satisfies the boundary condition of Problem P1, i.e., $\tilde{\mathbf{y}}(\mathbf{x}) = \mathbf{F}^*\mathbf{x}$ for all $\mathbf{x} \in \partial\mathcal{B}_0$, as well as the condition $\rho_0 \det \nabla\tilde{\mathbf{y}}(\mathbf{x}) = \tilde{\nu}(\mathbf{x})$, where $\tilde{\nu}(\mathbf{x})$ is given in (3.2), and suppose $\mathbf{y}(\mathbf{x}) \in \mathcal{A}_1$ is a function that also satisfies the boundary condition of Problem P1; we define, as usual, $\nu(\mathbf{x}) := \rho_0 \det \nabla\mathbf{y}(\mathbf{x})$. Then, it readily follows that

$$\begin{aligned}\int_{\mathcal{B}_0} \rho_0\Big(\hat{\varepsilon}\big(\nabla\mathbf{y}(\mathbf{x})\big) - \hat{\varepsilon}\big(\nabla\tilde{\mathbf{y}}(\mathbf{x})\big)\Big)\,dv &= \int_{\mathcal{B}_0} \rho_0\Big(\bar{\varepsilon}\big(\nu(\mathbf{x})\big) - \bar{\varepsilon}\big(\tilde{\nu}(\mathbf{x})\big)\Big)\,dv \\ &\geq \int_{\mathcal{B}_0} \rho_0 D_\nu\bar{\varepsilon}(\tilde{\nu}(\mathbf{x}))\big(\nu(\mathbf{x}) - \tilde{\nu}(\mathbf{x})\big)\,dv.\end{aligned} \tag{3.4}$$

But, because (2.4a) requires $D_\nu\bar{\varepsilon}(\tilde{\nu}(\mathbf{x})) = D_\nu\bar{\varepsilon}(\nu_1) = D_\nu\bar{\varepsilon}(\nu_2)$ for all $\mathbf{x} \in \mathcal{B}_0$, and both $\tilde{\mathbf{y}}(\mathbf{x})$ and $\mathbf{y}(\mathbf{x})$ satisfy the conditions for the validity of (1.6), we see that the integral on the right hand side above vanishes, which shows that $\tilde{\mathbf{y}}(\mathbf{x})$ is a minimizer of Problem P1. Our aim is to now exhibit such a minimizing field $\tilde{\mathbf{y}}(\mathbf{x})$. We first suppose that $\mathcal{B}_0$ is a ball and then consider the case when $\mathcal{B}_0$ is a body of arbitrary shape.

3.1 The case when $\mathcal{B}_0$ is a ball

It is convenient to begin with the special case when $\mathcal{B}_0$ is a ball of radius R_0, centered at the point $\mathbf{0} \in \mathbb{E}^3$: We denote this spherical domain as $\mathcal{B}_{R_0}$. Suppose $\mathcal{B}_{R_0}$ is decomposed into a set of concentric spherical shells $\mathcal{P}_{R_0}$ (with center $\mathbf{0}$)and a remainder $\mathcal{B}_{R_0} \setminus \mathcal{P}_{R_0}$, where $\operatorname{vol}\mathcal{P}_{R_0}$ is determined by (2.7) and

$$\mu = \frac{\nu^* - \nu_2}{\nu_1 - \nu_2} = \frac{\operatorname{vol}\mathcal{P}_{R_0}}{\operatorname{vol}\mathcal{B}_{R_0}}.$$

Theorem 3.1 *Define*

$$\tilde{\mathbf{y}}(\mathbf{x}) = \mathbf{F}^*\tilde{\mathbf{u}}(\mathbf{x}), \quad \tilde{\mathbf{u}}(\mathbf{x}) := r(R)\frac{\mathbf{x}}{R}, \quad R = |\mathbf{x}|, \quad R \in [0, R_0], \tag{3.5a}$$

where $r(R)$ *is given by*

$$\frac{4}{3}\pi r^3(R) := \int_{\mathcal{B}_R} \Big(\chi_{\mathcal{P}_{R_0}}(\mathbf{x})\frac{\nu_1}{\nu^*} + \big(1 - \chi_{\mathcal{P}_{R_0}}(\mathbf{x})\big)\frac{\nu_2}{\nu^*}\Big)\, dv, \quad R \in [0, R_0], \tag{3.5b}$$

with $\mathcal{B}_R$ *denoting the ball of radius* $R \in [0, R_0]$ *and* $\chi_{\mathcal{P}_{R_0}}(\cdot)$ *denoting the characteristic function for the set* $\mathcal{P}_{R_0} \subset \mathcal{B}_{R_0}$*. Then,* $\tilde{\mathbf{y}}(\cdot) \in W^{1,\infty}(\mathcal{B}_{R_0})$ *and is a minimizer for* Problem P1 *with* $\mathcal{B} = \mathcal{B}_{R_0}$*.*

Proof: From the definition of $r(R)$ it follows that

$$\frac{4\pi}{3}R^3\frac{\nu_2}{\nu^*} \geq \frac{4\pi}{3}r^3(R) \geq \frac{4\pi}{3}R^3\frac{\nu_1}{\nu^*} \quad \text{for } R \in [0, R_0]$$

and, hence,

$$\left(\frac{\nu_2}{\nu^*}\right)^{\frac{1}{3}} \geq \frac{r(R)}{R} \geq \left(\frac{\nu_1}{\nu^*}\right)^{\frac{1}{3}}. \tag{3.6}$$

Moreover, differentiating (3.5b) we obtain

$$4\pi r(R)^2 r'(R) = \int_{\partial\mathcal{B}_R} \Big(\chi_{\mathcal{P}_{R_0}}(\mathbf{x})\frac{\nu_1}{\nu^*} + \big(1 - \chi_{\mathcal{P}_{R_0}}(\mathbf{x})\big)\frac{\nu_2}{\nu^*}\Big)\, da \quad \text{for a.e. } R \in (0, R_0) \tag{3.7}$$

and, hence,

$$4\pi R^2 \frac{\nu_2}{\nu^*} \geq 4\pi r^2(R) r'(R) \geq 4\pi R^2 \frac{\nu_1}{\nu^*} \quad \text{for a.e. } R \in (0, R_0) \tag{3.8}$$

which, together with (3.6), yields

$$\frac{\nu_2}{\nu_1^{\frac{2}{3}} (\nu^*)^{\frac{1}{3}}} \geq r'(R) \geq \frac{\nu_1}{\nu_2^{\frac{2}{3}} (\nu^*)^{\frac{1}{3}}} \quad \text{for a.e. } R \in (0, R_0). \tag{3.9}$$

It then follows (e.g., using Lemma 4.1 in [2], (3.5a), (3.6) and (3.9)) that $\tilde{\mathbf{y}}(\cdot) \in W^{1,\infty}(\mathcal{B}_0)$ and that the distributional derivatives of $\tilde{\mathbf{u}}(\cdot)$ are given by

$$\nabla \tilde{\mathbf{u}}(\mathbf{x}) = r'(R) \left(\frac{\mathbf{x} \otimes \mathbf{x}}{|\mathbf{x}|^2} \right) + \frac{r(R)}{R} \left(\mathbf{1} - \frac{\mathbf{x} \otimes \mathbf{x}}{|\mathbf{x}|^2} \right). \tag{3.10}$$

Moreover,

$$\begin{aligned} \frac{4}{3}\pi r^3(R_0) &= \operatorname{vol} \mathcal{P}_{R_0} \frac{\nu_1}{\nu^*} + \big(\operatorname{vol} \mathcal{B}_{R_0} - \operatorname{vol} \mathcal{P}_{R_0} \big) \frac{\nu_2}{\nu^*} \\ &= \frac{\operatorname{vol} \mathcal{B}_{R_0}}{\nu^*} \big(\mu\nu_1 + (1-\mu)\nu_2 \big) = \operatorname{vol} \mathcal{B}_{R_0} = \frac{4}{3}\pi R_0^3, \end{aligned}$$

which shows that $r(R_0) = R_0$ and, accordingly, guarantees that $\tilde{\mathbf{y}}(\mathbf{x}) = \mathbf{F}^*\mathbf{x}$ for all $\mathbf{x} \in \partial \mathcal{B}_{R_0}$. In addition, note from (3.5a) and (3.10) that $\det \nabla \tilde{\mathbf{u}}(\mathbf{x}) = r'(R) \big(\frac{r(R)}{R} \big)^2$. Thus, by (3.7) we have

$$\det \nabla \tilde{\mathbf{u}}(\mathbf{x}) = \frac{1}{4\pi R^2} \int_{\partial \mathcal{B}_R} \Big(\chi_{\mathcal{P}_{R_0}}(\mathbf{x}) \frac{\nu_1}{\nu^*} + \big(1 - \chi_{\mathcal{P}_{R_0}}(\mathbf{x})\big) \frac{\nu_2}{\nu^*} \Big) \, da, \quad \text{for a.e. } R \in (0, R_0),$$

and we see, from $\tilde{\mathbf{y}}(\mathbf{x}) = \mathbf{F}^* \tilde{\mathbf{u}}(\mathbf{x})$ and $\nu^* = \nu_0 \det \mathbf{F}^*$, that

$$\nu_0 \det \nabla \tilde{\mathbf{y}}(\mathbf{x}) = \tilde{\nu}(\mathbf{x}) = \chi_{\mathcal{P}_{R_0}}(\mathbf{x})\nu_1 + \big(1 - \chi_{\mathcal{P}_{R_0}}(\mathbf{x})\big)\nu_2, \tag{3.11}$$

for all $\mathbf{x} \in \mathcal{B}_{R_0}$. Finally, we note that (3.9) also guarantees that the map $\tilde{\mathbf{y}}(\cdot)$ given by (3.5) is one-to-one on $\mathcal{B}_{R_0}$. It follows that $\tilde{\mathbf{y}}(\mathbf{x})$ of (3.5) belongs to the admissible set $\mathcal{A}_1$ and satisfies the boundary condition recorded in (1.4) for Problem P1 as well as (3.11), and these are sufficient for (3.4) and the argument of minimization following (3.4) to hold. The conclusion is that (3.5) defines a minimizer for Problem P1 when the region $\mathcal{B}_0$ is a ball of radius R_0 centered at $\mathbf{0} \in \mathbb{E}^3$. □

Remark 1: In a rough, but descriptive, sense the deformation gradient field $\nabla \tilde{\mathbf{y}}(\mathbf{x})$ associated with the minimizer (3.5) may be thought of as being composed of suitable combinations of linear transformation fields in Lin^+ that lie in the union of the surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$. It is noteworthy that more than two linear transformations, $\mathbf{F}_1 \in \mathcal{S}_1$ and $\mathbf{F}_2 \in \mathcal{S}_2$, are involved in the composition of this minimizing field. □

3.2 The case when $\mathcal{B}_0$ is a body of arbitrary shape

We now consider the case of a bounded, arbitrarily shaped reference configuration $\mathcal{B}_0$ and identify a minimizer $\check{\mathbf{y}}(\mathbf{x})$ for Problem P1. Our strategy is to use the minimizing solution (3.5) of Problem P1 for the case of a spherical reference domain of radius R_0 together with a scaling and covering technique suggested by Sivaloganathan and Spector [10][‖]. Because our approach will employ the inequality (3.4) in order to verify that a particular field is a minimizer, we need to identify a field $\check{\mathbf{y}}(\mathbf{x}) \in \mathcal{A}_1$ that satisfies the two conditions which are essential for (3.4) to hold. These are:

(i) The boundary condition $\check{\mathbf{y}}(\mathbf{x}) = \mathbf{F}^*\mathbf{x}$ for all $\mathbf{x} \in \partial\mathcal{B}_0$.

(ii) The 'convex combination' condition

$$\check{\nu}(\mathbf{x}) := \nu_0 \det \nabla\check{\mathbf{y}}(\mathbf{x}) = \chi_{\mathcal{P}_0}(\mathbf{x})\nu_1 + \big(1 - \chi_{\mathcal{P}_0}(\mathbf{x})\big)\nu_2 \tag{3.12a}$$

for all $\mathbf{x} \in \mathcal{B}_0$, where $\chi_{\mathcal{P}_0}(\cdot)$ is the characteristic function for the set $\mathcal{P}_0 \subset \mathcal{B}_0$ whose volume is determined by (2.7) and

$$\mu = \frac{\nu^* - \nu_2}{\nu_1 - \nu_2} = \frac{\operatorname{vol}\mathcal{P}_0}{\operatorname{vol}\mathcal{B}_0}. \tag{3.12b}$$

According to (3.2) and (3.3), this condition is equivalent to the condition that $\check{\nu}(\mathbf{x}) = \tilde{\nu}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{B}_0$, where $\tilde{\nu}(\mathbf{x})$ is a minimizer of Problem P2.

First, let us decompose $\mathcal{B}_0$ (up to volume measure zero) into the disjoint union of closed balls

$$\mathcal{B}_0 = \bigcup_{i=1}^{\infty} B_{\epsilon_i}(\mathbf{x}_i),$$

each centered at $\mathbf{x}_i \in \mathbb{E}^3$ and of radius $\epsilon_i > 0$[**]. Now, consider the field

[‖] See, also, Ball and Murat [3].

[**] See Corollary 2 to the Vitali Covering Theorem in [8].

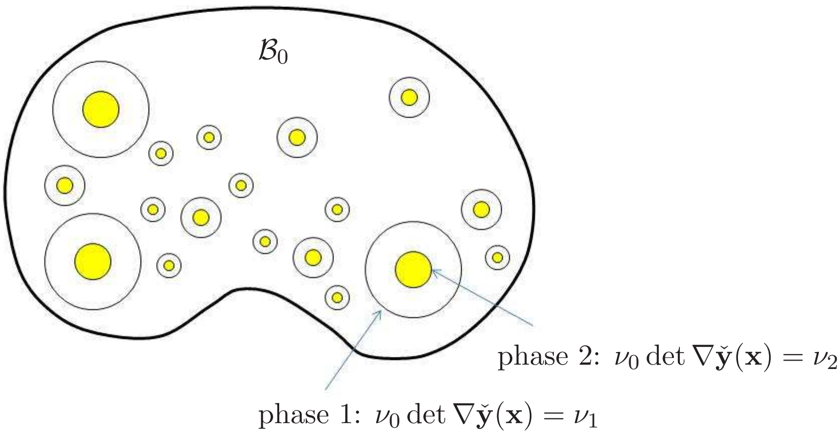

Figure 2: Scaling and covering of $\mathcal{B}_0$.

$$\check{\mathbf{y}}(\mathbf{x}) = \begin{cases} \frac{\epsilon_i}{R_0}\tilde{\mathbf{y}}(\mathbf{z})\big|_{\mathbf{z}=\frac{\mathbf{x}-\mathbf{x}_i}{\epsilon_i/R_0}} + \mathbf{F}^*\mathbf{x}_i & \forall\ \mathbf{x}\in B_{\epsilon_i}(\mathbf{x}_i), \\ \\ \mathbf{F}^*\mathbf{x} & \text{otherwise,} \end{cases} \tag{3.13}$$

where $\tilde{\mathbf{y}}(\cdot)$ is defined in (3.5). This decomposition and covering is sparsely illustrated in Figure 2.[††] Clearly, $\check{\mathbf{y}}(\cdot) \in W^{1,\infty}(\mathcal{B}_0)$ because $\tilde{\mathbf{y}}(\cdot) \in W^{1,\infty}(\mathcal{B}_0)$, and it follows from (3.13) that $\check{\mathbf{y}}(\mathbf{x}) = \mathbf{F}^*\mathbf{x}$ for all $\mathbf{x} \in \partial\mathcal{B}_0$, so that the condition (i) above holds. (Notice that by construction $\check{\mathbf{y}}(\mathbf{x}) = \mathbf{F}^*\mathbf{x}$ for all $\mathbf{x} \in \partial B_{\epsilon_i}(\mathbf{x}_i)$.) We also note that the construction (3.13) yields a mapping that is one-to-one almost everywhere.

A second elementary observation that follows from (3.13) is that

$$\nabla\check{\mathbf{y}}(\mathbf{x}) = \big(\nabla\tilde{\mathbf{y}}(\mathbf{z})\big)\big|_{\mathbf{z}=\frac{\mathbf{x}-\mathbf{x}_i}{\epsilon_i/R_0}} \quad \forall\ \mathbf{x}\in B_{\epsilon_i}(\mathbf{x}_i).$$

Thus, using the first of (3.11), we see that

$$\check{\nu}(\mathbf{x}) := \nu_0 \det \nabla\check{\mathbf{y}}(\mathbf{x}) = \tilde{\nu}(\mathbf{z})\big|_{\mathbf{z}=\frac{\mathbf{x}-\mathbf{x}_i}{\epsilon_i/R_0}} \quad \forall\ \mathbf{x}\in B_{\epsilon_i}(\mathbf{x}_i),$$

[††] Here, balls containing only two separated, but connected, phase regions are shown. However, balls having a multiple concentric shell structure, each containing deformation gradients from the surfaces $\mathcal{S}_1$ or $\mathcal{S}_2$ which define the two phases, can be envisioned.

which, with the second of (3.11), allows the conclusion

$$\check{\nu}(\mathbf{x}) = \big(\chi_{\mathcal{P}_{R_0}}(\mathbf{z})\nu_1 + \big(1 - \chi_{\mathcal{P}_{R_0}}(\mathbf{z})\big)\nu_2\big)\big|_{\mathbf{z}=\frac{\mathbf{x}-\mathbf{x}_i}{\epsilon_i/R_0}} \quad \forall\, \mathbf{x}{\in}B_{\epsilon_i}(\mathbf{x}_i).$$

Here, to be clear in our interpretation, we observe that

$$\chi_{P_{\epsilon_i}(\mathbf{x}_i)}(\mathbf{x}) := \chi_{\mathcal{P}_{R_0}}\left(\frac{\mathbf{x}-\mathbf{x}_i}{\epsilon_i/R_0}\right) \quad \forall\, \mathbf{x}{\in}B_{\epsilon_i}(\mathbf{x}_i)$$

denotes the characteristic function for the set $P_{\epsilon_i}(\mathbf{x}_i){\subset}B_{\epsilon_i}(\mathbf{x}_i)$ whose volume is determined by (2.7) and

$$\mu = \frac{\nu^* - \nu_2}{\nu_1 - \nu_2} = \frac{\operatorname{vol} P_{\epsilon_i}(\mathbf{x}_i)}{\operatorname{vol} B_{\epsilon_i}(\mathbf{x}_i)}, \tag{3.15a}$$

and, accordingly, we may write

$$\check{\nu}(\mathbf{x}) = \chi_{P_{\epsilon_i}(\mathbf{x}_i)}(\mathbf{x})\nu_1 + \big(1 - \chi_{P_{\epsilon_i}(\mathbf{x}_i)}(\mathbf{x})\big)\nu_2 \quad \forall\, \mathbf{x}{\in}B_{\epsilon_i}(\mathbf{x}_i). \tag{3.15b}$$

However, defining

$$\operatorname{vol}\mathcal{P}_0 := \sum_{i=1}^{\infty} \operatorname{vol} P_{\epsilon_i}(\mathbf{x}_i),$$

observing

$$\operatorname{vol}\mathcal{B}_0 = \sum_{i=1}^{\infty} \operatorname{vol} B_{\epsilon_i}(\mathbf{x}_i),$$

and noting, from (3.15a), that the ratio $\operatorname{vol} P_{\epsilon_i}(\mathbf{x}_i)/\operatorname{vol} B_{\epsilon_i}(\mathbf{x}_i)$ is fixed, independent of the index i, we see from (3.15) that (3.12) is valid for the field introduced in (3.13). Thus, the condition (ii) above holds and, because condition (i) also holds, we thus obtain the following result:

Theorem 3.2 *Let $\mathcal{B}_0$ be an arbitrary domain and let $\check{\mathbf{y}}(\cdot)$ be defined by* (3.13)*. Then $\check{\mathbf{y}}(\cdot)$ is a minimizer for* Problem P1 *(given by* (1.4)*) on $\mathcal{B}_0$.*

The total internal energy in $\mathcal{B}_0$ corresponding to the minimizing field (3.13) for Problem P1 and the total internal energy in the ball $\mathcal{B}_{R_0}$ corresponding to the minimizing field (3.5) for Problem P1 are equal to one another if the radius R_0 of the ball is chosen such that the volumes are equal, i.e., $\operatorname{vol}\mathcal{B}_{R_0} = \frac{4}{3}\pi R_0^3 = \operatorname{vol}\mathcal{B}_0$. The general relationship between the two energies is given by

$$\begin{aligned}\int_{\mathcal{B}_0} \rho_0 \hat{\varepsilon}\big(\nabla \check{\mathbf{y}}(\mathbf{x})\big)\, dv &= \sum_{i=1}^{\infty} \int_{B_{\epsilon_i}(\mathbf{x}_i)} \rho \hat{\varepsilon}\big(\nabla \check{\mathbf{y}}(\mathbf{x})\big)\, dv \\ &= \sum_{i=1}^{\infty} \Big(\frac{\epsilon_i}{R_0}\Big)^3 \int_{\mathcal{B}_{R_0}} \rho_0 \hat{\varepsilon}\big(\nabla \tilde{\mathbf{y}}(\mathbf{z})\big)\, dv \\ &= \frac{\operatorname{vol}\mathcal{B}_0}{\operatorname{vol}\mathcal{B}_{R_0}} \int_{\mathcal{B}_{R_0}} \rho_0 \hat{\varepsilon}\big(\nabla \tilde{\mathbf{y}}(\mathbf{z})\big)\, dv. \end{aligned} \tag{3.17}$$

Remark 2: The field $\check{\mathbf{y}}(\cdot)$ of (3.13) is based on the minimizer (3.5) for $\mathcal{B}_{R_0}$, the ball of radius R_0 centered at $\mathbf{0} \in \mathbb{E}^3$; by covering the arbitrary region $\mathcal{B}_0$ with a countable number of balls, (3.5) was used above to construct a minimizer for Problem P1 for the arbitrary region $\mathcal{B}_0$. Of course, $\mathcal{B}_{R_0}$, itself, qualifies to be chosen as the 'arbitrary region' $\mathcal{B}_0$ and in that case by a covering and scaling strategy the field $\check{\mathbf{y}}(\cdot)$ of (3.13) readily can be used to identify another minimizer, in addition to (3.5), for Problem P1 for the ball $\mathcal{B}_{R_0}$. This minimizer will also consist of a countable number of smaller balls within $\mathcal{B}_{R_0}$, and the resulting "mixture" of deformation gradients from the union of the surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ will be finer than what it was for the minimizer (3.5). This iterated process of covering and scaling sub-balls with smaller balls which support a field based upon the last iteration can be continued in order to produce a minimizer $\tilde{\mathbf{y}}(\cdot) \in \mathcal{A}_1$ for Problem P1 for the region $\mathcal{B}_{R_0}$ which has as fine a "mixture" of deformation gradients $\mathbf{F}_1 \in \mathcal{S}_1$ and $\mathbf{F}_2 \in \mathcal{S}_2$ as one wishes. Of course, the minimal total internal energy for the Problem P1 is the same for each iteration in this sequence.

After a sufficient number of iterations, a suitably refined minimizer $\tilde{\mathbf{y}}(\cdot) \in \mathcal{A}_1$ for Problem P1 for the ball $\mathcal{B}_{R_0}$ is obtained. This can be used in conjunction with (3.13) and the process of covering and scaling to define a new minimizer for Problem P1 for the *arbitrary* domain $\mathcal{B}_0$ just as was described above. This minimizer will consist of a "mixture" of deformation gradients $\mathbf{F}_1 \in \mathcal{S}_1$ and $\mathbf{F}_2 \in \mathcal{S}_2$, and the iteration procedure will produce a "mixture" as a minimizing field that is as fine as one pleases. □

4 A generalized minimization problem

Intuitively, it would seem that for an elastic fluid the solutions to the minimization Problem P2 should relate well with the solutions to a generalized version of the minimization Problem

P1, wherein the form of the prescribed boundary placement is arbitrarily specified, to the extent that the deformed volume is the same as that given in Problem P2. This **Generalized Problem GP1** may be stated as:

$$\underset{\mathbf{y}(\cdot)\in\mathcal{A}_1}{\mathsf{minimize}} \int_{\mathcal{B}_0} \rho_0 \hat{\varepsilon}(\nabla \mathbf{y}(\mathbf{x}))\, dv, \quad \text{with } \mathbf{y}(\mathbf{x}) = \mathbf{y}^*(\mathbf{x}) \ \forall \mathbf{x} \in \partial\mathcal{B}_0, \tag{4.1a}$$

where the boundary placement $\mathbf{y}^*(\mathbf{x})$ is given so that the volume of $\mathcal{B} = \mathbf{y}(\mathcal{B}_0)$ is the same as that prescribed in (1.5) and (1.7), i.e.,

$$\operatorname{vol}\mathcal{B} = \nu^* \rho_0 \operatorname{vol}\mathcal{B}_0. \tag{4.1b}$$

If $\vec{\mathbf{y}}(\mathbf{x})$ is a minimizer of the Generalized Problem GP1 (4.1), then the range of $\nabla\vec{\mathbf{y}}(\cdot)$ can only correspond to rank 1 convex points of $\hat{\varepsilon}(\cdot)$. Consequently, because of Theorem 2.1, the range of $\nu_0 \det \nabla\vec{\mathbf{y}}(\cdot)$ can only correspond to convex points of $\overline{\varepsilon}(\cdot)$. Thus, if $\tilde{\nu}(\mathbf{x})$ is a minimizer of Problem P2 we see, using (1.1), that

$$\int_{\mathcal{B}_0} \rho\hat{\varepsilon}\big(\nabla\vec{\mathbf{y}}(\mathbf{x})\big)\, dv = \int_{\mathcal{B}_0} \rho_0 \overline{\varepsilon}\big(\nu_0 \det \nabla\vec{\mathbf{y}}(\mathbf{x})\big)\, dv \geq \int_{\mathcal{B}_0} \rho_0 \overline{\varepsilon}\big(\tilde{\nu}(\mathbf{x})\big)\, dv.$$

However, from our earlier constructions in Section 3 we know that

$$\int_{\mathcal{B}_0} \rho\hat{\varepsilon}\big(\nabla\tilde{\mathbf{y}}(\mathbf{x})\big)\, dv = \int_{\mathcal{B}_0} \rho_0 \overline{\varepsilon}\big(\tilde{\nu}(\mathbf{x})\big)\, dv,$$

where $\tilde{\mathbf{y}}(\mathbf{x})$ is a minimizer of the corresponding Problem P1, which satisfies the special boundary condition $\tilde{\mathbf{y}}(\mathbf{x}) = \mathbf{F}^*\mathbf{x}$ for all $\mathbf{x} \in \partial\mathcal{B}_0$. Thus,

$$\int_{\mathcal{B}_0} \rho\hat{\varepsilon}\big(\nabla\vec{\mathbf{y}}(\mathbf{x})\big)\, dv \geq \int_{\mathcal{B}_0} \rho\hat{\varepsilon}\big(\nabla\tilde{\mathbf{y}}(\mathbf{x})\big)\, dv,$$

and we see that for equal distorted volumes the minimum total internal energy of the Generalized Problem GP1 cannot be less than the minimum total internal energy of the Problem P1.

Although the boundary conditions for the Generalized Problem GP1 and the Problem P1 are different in detail, it would seem to be the nature of an elastic fluid and its affinity towards the rearrangement of its particles to accommodate a minimal equilibrium state that the inequalities above should be equalities; after all, the total volumes of the distorted configurations are equal. This can be shown with the aid of a theorem of Dacorogna and Moser [7], which guarantees the following:

Proposition 1 *Let $\Omega \subset \mathbb{E}^3$ be a bounded, open, connected set with C^1 boundary $\partial\Omega$ and let $\boldsymbol{\phi}_0(\cdot)$ be an orientation preserving C^2-diffeomorphism of Ω*[‡‡]*. Suppose further that* $\mathrm{vol}\,(\Omega) = \mathrm{vol}\,(\boldsymbol{\phi}_0(\Omega))$*. Then there exists a C^1-diffeomorphism $\boldsymbol{\phi}(\cdot)$ of Ω such that*

$$\det \nabla\boldsymbol{\phi}(\mathbf{z}) = 1 \quad \forall\, \mathbf{z} \in \Omega,$$

$$\boldsymbol{\phi}(\mathbf{z}) = \boldsymbol{\phi}_0(\mathbf{z}) \quad \forall\, \mathbf{z} \in \partial\Omega. \tag{4.2}$$

Proof: Apply Theorem 5 (and Remark (ii)) in [7] on $\Omega_0 = \boldsymbol{\phi}_0(\Omega)$ with $k = 1, g \equiv 1, f = \det\left(\nabla\phi_0^{-1}\right)$ to obtain a C^1-diffeomorphism $\tilde{\boldsymbol{\phi}}(\cdot)$ of Ω_0 satisfying $\det(\nabla\tilde{\boldsymbol{\phi}}(\mathbf{y})) = f(\mathbf{y})$ for $\mathbf{y} \in \Omega_0$ with $\tilde{\boldsymbol{\phi}}(\mathbf{y}) = \mathbf{y}$ for all $\mathbf{y} \in \partial\Omega_0$. Next, set $\boldsymbol{\phi}(\mathbf{z}) = \tilde{\boldsymbol{\phi}} \circ \boldsymbol{\phi}_0(\mathbf{z})$ for all $\mathbf{z} \in \Omega$ to obtain the result. □

In applying Proposition 1, we make the identification $\Omega := \left\{\mathbf{z} \in \mathbb{E}^3 \,|\, \mathbf{z} = \mathbf{F}^*\mathbf{x}, \mathbf{x} \in \mathcal{B}_0\right\}$, where $\mathbf{F}^* \in \mathrm{Lin}^+$, and assume that $\mathcal{B}_0$ has a class C^1 boundary $\partial\mathcal{B}_0$. In addition, we take

$$\boldsymbol{\phi}_0(\mathbf{z}) := \mathbf{y}_E^*(\mathbf{x})|_{\mathbf{x}=\mathbf{F}^{*-1}\mathbf{z}}, \quad \mathbf{z} \in \bar{\Omega}, \tag{4.3}$$

where $\mathbf{y}_E^*(\cdot)$ is a C^2-diffeomorphism of $\mathcal{B}_0$ which satisfies $\mathbf{y}_E^*(\mathbf{x}) = \mathbf{y}^*(\mathbf{x})$ for all $\mathbf{x} \in \partial\mathcal{B}_0$. (Of course, the boundary placement $\mathbf{y}^*(\mathbf{x})$ of the Generalized Problem GP1, introduced in (4.1), must be assumed to be sufficiently regular.)

Case 1: $\nu^* := \nu_0 \det \mathbf{F}^*$ is a convex point of $\bar{\varepsilon}(\cdot)$

Suppose, for this case, that $\vec{\mathbf{y}}(\mathbf{x})$ satisfies the following two conditions:

(i) $\vec{\mathbf{y}}(\mathbf{x}) = \mathbf{y}^*(\mathbf{x})$ for all $\mathbf{x} \in \partial\mathcal{B}_0$, where $\mathbf{y}^*(\mathbf{x})$ satisfies (4.1b);

(ii) $\nu_0 \det \nabla\vec{\mathbf{y}}(\mathbf{x}) = \nu^*$ for all $\mathbf{x} \in \mathcal{B}_0$, i.e., the range of the deformation gradient field, $\nabla\vec{\mathbf{y}}(\mathbf{x})$, is required to lie in the surface $\mathcal{S}^* := \{\mathbf{F} \in \mathrm{Lin}^+ | \;\; \nu_0 \det \mathbf{F} = \nu^*\}$.

Then, it follows by an argument similar to that used following (3.1) that $\vec{\mathbf{y}}(\mathbf{x})$ is a minimizer of the Generalized Problem GP1. Moreover, the inequalities above become equalities because in this case a minimizer of Problem P2 is known to be $\tilde{\nu}(\mathbf{x}) = \nu^*$.

To see that such a field $\vec{\mathbf{y}}(\mathbf{x})$ exists, we first recall, from the argument following (3.1), that in this case $\tilde{\mathbf{y}}(\mathbf{x}) = \mathbf{F}^*\mathbf{x}$ is a minimizer of Problem P1, and observe that $\tilde{\mathbf{y}}(\cdot)$ maps $\mathcal{B}_0 \to \Omega$. Now consider the composition

[‡‡] Given two open sets $\Omega_1, \Omega_2 \subset \mathbb{E}^n$. A bijective map $\boldsymbol{\psi}(\cdot) : \bar{\Omega}_1 \to \bar{\Omega}_2$ is said to be a C^r-diffeomorphism of Ω_1, $r \geq 1$, if both $\boldsymbol{\psi}(\cdot) : \Omega_1 \to \Omega_2$ and $\boldsymbol{\psi}^{-1}(\cdot) : \Omega_2 \to \Omega_1$ satisfy $\boldsymbol{\psi}(\cdot) \in C^r(\bar{\Omega}_1)$ and $\boldsymbol{\psi}^{-1}(\cdot) \in C^r(\bar{\Omega}_2)$. We say that $\boldsymbol{\psi}(\cdot)$ is orientation preserving if $\det \nabla\boldsymbol{\psi}(\mathbf{x}) > 0$ for all $\mathbf{x} \in \bar{\Omega}_1$.

$$\vec{\mathbf{y}}(\mathbf{x}) := \boldsymbol{\phi} \circ \tilde{\mathbf{y}}(\mathbf{x}) \quad \forall\, \mathbf{x} \in \bar{\mathcal{B}}_0. \tag{4.4}$$

Clearly, $\vec{\mathbf{y}}(\cdot) \in C^1(\bar{\mathcal{B}}_0)$ and because of the boundary condition in (4.2) we have

$$\vec{\mathbf{y}}(\mathbf{x}) = \boldsymbol{\phi}(\mathbf{F}^*\mathbf{x}) = \boldsymbol{\phi}_0(\mathbf{F}^*\mathbf{x}) = \mathbf{y}_E^*(\mathbf{x}) = \mathbf{y}^*(\mathbf{x})$$

for all $\mathbf{x} \in \partial\mathcal{B}_0$. Moreover, because of the first of (4.2) and the chain rule, we see, using (4.4), that

$$\det \nabla\vec{\mathbf{y}}(\mathbf{x}) = \det \nabla\tilde{\mathbf{y}}(\mathbf{x}) = \det \mathbf{F}^* \quad \forall\, \mathbf{x} \in \mathcal{B}_0,$$

which means that $\nu_0 \det \nabla\vec{\mathbf{y}}(\mathbf{x}) = \nu^*$ for all $\mathbf{x} \in \mathcal{B}_0$. Thus, the conditions (i) and (ii) noted above hold and we may conclude that for this case the field $\vec{\mathbf{y}}(\cdot)$ defined through the composition (4.4) is a minimizer of the Generalized Problem GP1.

Case 2: $\nu^* := \nu_0 \det \mathbf{F}^*$ is a non-convex point of $\bar{\varepsilon}(\cdot)$

Suppose, for this case, that $\vec{\mathbf{y}}(\mathbf{x})$ satisfies the following two conditions:

(i) $\vec{\mathbf{y}}(\mathbf{x}) = \mathbf{y}^*(\mathbf{x})$ for all $\mathbf{x} \in \partial\mathcal{B}_0$, where $\mathbf{y}^*(\mathbf{x})$ satisfies (4.1b);

(ii) $\nu_0 \det \nabla\vec{\mathbf{y}}(\mathbf{x}) = \tilde{\nu}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{B}_0$, where $\tilde{\nu}(\mathbf{x})$, given by (3.2) and (3.3), is a minimizer of Problem P2, i.e., the range of the deformation gradient field, $\nabla\vec{\mathbf{y}}(\mathbf{x})$, is required to lie in the union of the surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$.

Then, again, it can be argued, as in (3.4), that $\vec{\mathbf{y}}(\mathbf{x})$ is a minimizer of the Generalized Problem GP1 and it follows that the inequalities above become equalities.

To see that such a field $\vec{\mathbf{y}}(\mathbf{x})$ exists, we first recall that in this case any minimizer $\check{\mathbf{y}}(\mathbf{x})$ of Problem P1, constructed from (3.13) or by an iteration as described in Remark 2, belongs to $W^{1,\infty}(\mathcal{B}_0)$, satisfies $\check{\mathbf{y}}(\mathbf{x}) = \mathbf{F}^*\mathbf{x}$ for all $\mathbf{x} \in \partial\mathcal{B}_0$ and maps $\bar{\mathcal{B}}_0 \to \bar{\Omega}$. Now, consider the composition

$$\vec{\mathbf{y}}(\mathbf{x}) := \boldsymbol{\phi} \circ \check{\mathbf{y}}(\mathbf{x}) \quad \forall\, \mathbf{x} \in \bar{\mathcal{B}}_0, \tag{4.5}$$

and observe that $\vec{\mathbf{y}}(\cdot) \in W^{1,\infty}(\mathcal{B}_0)$. In addition, because of the second of (4.2), we again have

$$\vec{\mathbf{y}}(\mathbf{x}) = \boldsymbol{\phi}(\mathbf{F}^*\mathbf{x}) = \boldsymbol{\phi}_0(\mathbf{F}^*\mathbf{x}) = \mathbf{y}_E^*(\mathbf{x}) = \mathbf{y}^*(\mathbf{x})$$

for all $\mathbf{x} \in \partial\mathcal{B}_0$. Thus, $\vec{\mathbf{y}}(\cdot) \in \mathcal{A}_1$ and satisfies the boundary condition recorded in (4.1a). Moreover, by the chain rule and the first of (4.2), we see, using (4.5), that

$$\det \nabla\vec{\mathbf{y}}(\mathbf{x}) = \det \nabla\check{\mathbf{y}}(\mathbf{x}) \quad \forall\, \mathbf{x} \in \mathcal{B}_0,$$

which, according to (3.12b), shows that $\nu_0 \det \nabla\vec{\mathbf{y}}(\mathbf{x}) = \tilde{\nu}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{B}_0$. Thus, conditions (i) and (ii), above, hold and we may thus conclude the following result:

Theorem 4.1 *Let $\mathcal{B}$ be a bounded, connected open set with C^1 boundary. Suppose that the boundary data $\mathbf{y}^*(\cdot)$ given in the* Generalized Problem GP1 *(see* (4.1)*) is the restriction to $\partial\mathcal{B}_0$ of an orientation preserving, C^2-diffeomorphism of $\mathcal{B}_0$. Let $\bar{\mathbf{y}}(\cdot)$ be a minimizer of* Problem P1 *for some $\mathbf{F}^*$ such that* (4.1b) *is satisfied (i.e., $\bar{\mathbf{y}}(\cdot) := \tilde{\mathbf{y}}(\cdot)$ for $\mathbf{F}^*$ as in* Case 1 *or $\bar{\mathbf{y}}(\cdot) := \check{\mathbf{y}}(\cdot)$ for $\mathbf{F}^*$ as in* Case 2*, above). Let $\boldsymbol{\phi}(\cdot)$ be the diffeomorphism given by* Proposition 1 *and* (4.3)*. Then $\vec{\mathbf{y}}(\cdot) = \boldsymbol{\phi} \circ \bar{\mathbf{y}}(\cdot)$ (which is one-to-one almost everywhere) is a minimizer for the* Generalized Problem GP1*.*

Clearly, the arguments used in the above theorem also show that the minimizers we construct for the Generalized Problem GP1 are also minimizers for the corresponding problem in which we replace the boundary condition in (4.1) by $\mathbf{y}(\mathbf{x}) = \mathbf{y}^{**}(\mathbf{x})$ for $\mathbf{x} \in \partial\mathcal{B}_0$ where $\mathbf{y}^{**}(\cdot) \in C^2(\bar{\mathcal{B}}_0)$ is any C^2-diffeomorphism of $\mathcal{B}_0$ such that $\mathbf{y}^{**}(\mathcal{B}_0) = \mathbf{y}^*(\mathcal{B}_0)$.

Remark 3: In both cases above where ν^* is either a convex or a non-convex point of $\overline{\varepsilon}(\cdot)$, it is straightforward to see that a minimizer $\vec{\mathbf{y}}(\mathbf{x})$ of the Generalized Problem GP1 satisfies the weak forms of the **equilibrium equation** as well as the **energy-momentum equation**. These are given, respectively, by

$$\int_{\mathcal{B}_0} \rho_0 D_{\mathbf{F}}\hat{\varepsilon}\big(\nabla\vec{\mathbf{y}}(\mathbf{x})\big) \cdot \nabla\mathbf{u}(\mathbf{x})\, dv = 0 \quad \forall\, \mathbf{u}(\cdot) \in C_0^1(\mathcal{B}_0) \tag{4.6a}$$

and

$$\int_{\mathcal{B}_0} \rho_0 \Big(\hat{\varepsilon}\big(\nabla\vec{\mathbf{y}}(\mathbf{x})\big)\mathbf{1} - \nabla\vec{\mathbf{y}}(\mathbf{x})^{\mathrm{T}} D_{\mathbf{F}}\hat{\varepsilon}\big(\nabla\vec{\mathbf{y}}(\mathbf{x})\big)\Big) \cdot \nabla\mathbf{u}(\mathbf{x})\, dv = 0 \quad \forall\, \mathbf{u}(\cdot) \in C_0^1(\mathcal{B}_0). \tag{4.6b}$$

To see that (4.6) holds, we first recall (1.1) and (1.2) and re-structure the integrands above using the identities

$$\rho_0 D_{\mathbf{F}}\hat{\varepsilon}(\nabla\vec{\mathbf{y}}(\mathbf{x})) = D_\nu\overline{\varepsilon}(\tilde{\nu}(\mathbf{x}))\, \mathrm{cof}\, \nabla\vec{\mathbf{y}}(\mathbf{x}) \tag{4.7a}$$

and

$$\rho_0\Big(\hat{\varepsilon}\big(\nabla\vec{\mathbf{y}}(\mathbf{x})\big)\mathbf{1}-\nabla\vec{\mathbf{y}}(\mathbf{x})^{\mathrm{T}}D_{\mathbf{F}}\hat{\varepsilon}\big(\nabla\vec{\mathbf{y}}(\mathbf{x})\big)\Big)$$
$$=\rho_0\overline{\varepsilon}\big(\tilde{\nu}(\mathbf{x})\big)\mathbf{1}-D_{\nu}\overline{\varepsilon}\big(\tilde{\nu}(\mathbf{x})\big)\nabla\vec{\mathbf{y}}(\mathbf{x})^{\mathrm{T}}\mathrm{cof}\,\nabla\vec{\mathbf{y}}(\mathbf{x})$$
$$=\rho_0\Big(\overline{\varepsilon}\big(\tilde{\nu}(\mathbf{x})\big)-D_{\nu}\overline{\varepsilon}\big(\tilde{\nu}(\mathbf{x})\big)\tilde{\nu}(\mathbf{x})\Big)\mathbf{1}. \tag{4.7b}$$

Here, we have used the notation $\tilde{\nu}(\mathbf{x}) := \nu_0 \det\nabla\vec{\mathbf{y}}(\mathbf{x})$, where $\vec{\mathbf{y}}(\mathbf{x})$ is a minimizer for the Generalized Problem GP1 in either of Case 1 and Case 2 covered above in this section.

Now, to confirm that (4.6a) holds we observe, regardless of the assigned value ν^*, that the minimizer $\vec{\mathbf{y}}(\cdot)\in\mathcal{A}_1$ satisfies $D_{\nu}\overline{\varepsilon}\big(\tilde{\nu}(\mathbf{x})\big)=\text{const.}$ Then, using (4.7a)and the divergence theorem, we have

$$\int_{\mathcal{B}_0}\rho_0 D_{\mathbf{F}}\hat{\varepsilon}\big(\nabla\vec{\mathbf{y}}(\mathbf{x})\big)\cdot\nabla\mathbf{u}(\mathbf{x})\,dv=\text{const.}\int_{\mathcal{B}_0}\mathrm{cof}\,\nabla\vec{\mathbf{y}}(\mathbf{x})\cdot\nabla\mathbf{u}(\mathbf{x})\,dv$$
$$=\text{const.}\int_{\mathcal{B}_0}\mathrm{div}\,\Big(\big(\mathrm{cof}\,\nabla\vec{\mathbf{y}}(\mathbf{x})\big)^{\mathrm{T}}\,\mathbf{u}(\mathbf{x})\Big)\,dv=0$$

for all $\mathbf{u}(\cdot)\in C_0^1(\mathcal{B}_0)$, which yields (4.6a) (see also expression (3.7) in Lemma 3.3 of [1]).

Similarly, to confirm (4.6b) we observe, regardless of the assigned value ν^*, that the minimizer $\vec{\mathbf{y}}(\cdot)\in\mathcal{A}_1$ satisfies $\rho_0\Big(\overline{\varepsilon}\big(\tilde{\nu}(\mathbf{x})\big)-D_{\nu}\overline{\varepsilon}\big(\tilde{\nu}(\mathbf{x})\big)\tilde{\nu}(\mathbf{x})\Big)=\text{const.}$ Then, using (4.7b) and the divergence theorem, we have

$$\int_{\mathcal{B}_0}\rho_0\Big(\hat{\varepsilon}\big(\nabla\vec{\mathbf{y}}(\mathbf{x})\big)\mathbf{1}-\nabla\vec{\mathbf{y}}(\mathbf{x})^{\mathrm{T}}D_{\mathbf{F}}\hat{\varepsilon}\big(\nabla\vec{\mathbf{y}}(\mathbf{x})\big)\Big)\cdot\nabla\mathbf{u}(\mathbf{x})\,dv$$
$$=\text{const.}\int_{\mathcal{B}_0}\mathrm{div}\,\mathbf{u}(\mathbf{x})\,dv=0\quad\forall\,\mathbf{u}(\cdot)\in C_0^1(\mathcal{B}_0),$$

which yields (4.6b). □

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References

[1] Ball, J.M., Constitutive inequalities and existence theorems in nonlinear elastostatics. *Nonlinear Analysis and Mechanics, Heriot-Watt Symposium, Vol. I.* Pitman, Ed: R.J.Knops, 1977.

[2] Ball, J.M., Discontinuous equilibrium solutions and cavitation in nonlinear elasticity, *Phil. Trans. R. Soc. Lond.*, A **306**, 557–611, (1982).

[3] Ball, J.M. & Murat, F., $W^{1,p}$-quasiconvexity and variational problems for multiple integrals. *J. Funct. Anal.*, **58**, 225-253, (1984).

[4] Cellini, A. & Zagatti, S., An existence result in a problem of the vectorial case of the calculus of variations. *SIAM J. Control Optim.*, **33**, 960-970, (1995).

[5] Cupini, G. & Mascolo, E., Existence of minimizers for polyconvex and nonpolyconvex problems. *SIAM J. Control Optim.*, **44**, 1370-1390, (2005).

[6] Dacorogna, B., *Direct Methods in the Calculus of Variations.* New York, Springer Science+Business Media, LLC, 2007.

[7] Dacorogna, B. & Moser, J., On a partial differential equation involving the Jacobian determinant. *Ann. Inst. Henri Poincaré*, **7**, 1-26, (1990).

[8] Evans, L.C. & Gariepy, R.F., *Measure Theory and Fine Properties of Functions.* CRC, 1991.

[9] Mizel, V.J., On the ubiquity of fracture in nonlinear elasticity. *J. Elasticity*, **52**, 257-266, (1999).

[10] Sivaloganathan, J. & Spector, S.J., A construction of infinitely many singular weak solutions to the equations of nonlinear elasticity. *Proc. R. Soc. Ed.*, **132**A, 985-992, (2002).