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On homotopy conditions and the existence of multiple equilibria in finite elasticity

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In this paper we study homotopy classes of deformations and their properties under weak convergence. As an application, we give an analytic proof (in two and three dimensions) of the existence of infinitely many local minimisers for a displacement boundary-value problem from finite elasticity, posed on a nonconvex domain, under the constitutive assumption of polyconvexity.

1. Introduction

The paper is motivated by a well-known (two-dimensional) heuristic example due to Fritz John (see e.g. [**11**]). Let the annulus

$$A = \{x \in \mathbb{R}^2 : a < |x| = (x^2 + y^2)^{\frac{1}{2}} < b\} \tag{1.1}$$

be the region occupied by a nonlinearly elastic material in its reference state and consider maps (deformations) $\boldsymbol{u}: A \to \mathbb{R}^2$ satisfying the boundary conditions $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{x}$ for all $\boldsymbol{x} \in \partial A$ and

$$\det \nabla \boldsymbol{u}(\boldsymbol{x}) > 0 \quad \text{for } \boldsymbol{x} \in \Omega. \tag{1.2}$$

Under the assumption that the material is hyperelastic (and under zero body forces) the energy stored by such a deformation is given by

$$E(\boldsymbol{u}) = \int_\Omega W(\nabla \boldsymbol{u}(\boldsymbol{x}))\, dx, \tag{1.3}$$

where $W: M_+^{2\times 2} \to \mathbb{R}^+$ is the stored energy function and $M_+^{2\times 2}$ denotes the space of 2×2 matrices with positive determinant (see e.g. [**6, 14, 20**]).

Heuristically, one expects multiple equilibria in such a displacement boundary-value problem corresponding, for example, to deformations of A which rotate the inner (or equivalently the outer) boundary by an integer multiple of 2π: for example, consider the maps $\boldsymbol{u}, \boldsymbol{v}$ in Figure 1.1, where we have indicated the image of a radial line under the map. Though heuristically plausible as an argument for multiple equilibria, there has to our knowledge been no analytic proof given of the existence of these multiple equilibria. In Theorem 3.1, we give conditions under which these

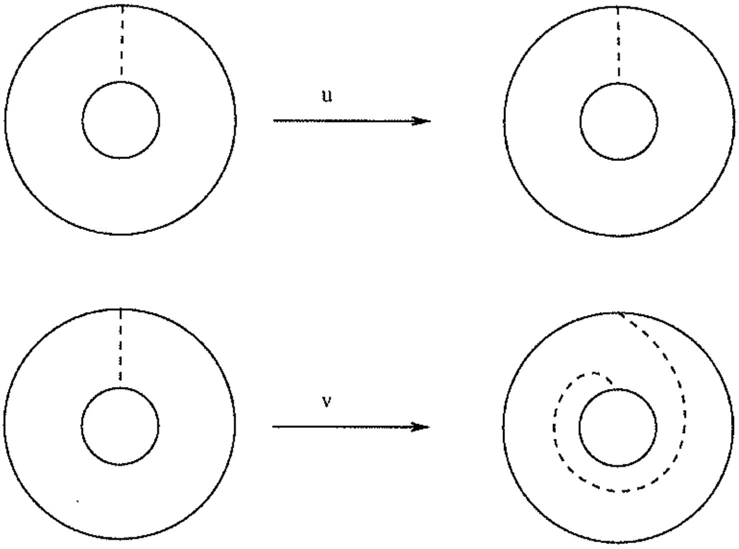

Figure 1.1

equilibria exist and show that they are all local minimisers of (1.3) (see Remark 3.2). To do this, we express the condition that the image of a radial line in A under a deformation winds n times around the origin as an analytic condition, using the classical notion of winding number (see Section 2 and Definition 2.8), and prove that this is preserved under sequential weak convergence in $W^{1,p}(A)$ for $p \geqq 1$. This enables us, for each $N \in \mathbb{N}$, to prove the existence of minimisers for (1.3) by the direct method of the calculation of variations in a class $\mathscr{A}^N$ of maps that winds almost every radial line in the annulus N-times around the origin (see Theorem 3.1).

In Lemma 3.6, we give an estimate for $\inf_{u \in \mathscr{A}^N} E(u)$ in terms of the winding number N. Once the existence of a minimiser u^N of E in $\mathscr{A}^N$ is established for each $N \in \mathbb{N}$, it is straightforward to argue that u^N is also a strong local minimiser of E in a class of deformations with no winding number constraint (see Remark 3.2). It also follows that if u^N is a strict local minimiser of E in $\mathscr{A}^N$, then u^N is radially symmetric (see Remark 3.5).

In Section 2, we review some basic results concerning the winding number of a curve in the plane, and apply these results to obtain an analytic condition on deformation of A which is preserved under weak convergence (see Definition 2.8). In Section 3, we apply this in two dimensions to obtain the existence of minimisers of E on $\mathscr{A}^N$ for each $N \in \mathbb{N}$. In Section 5, we give corresponding results for a three-dimensional problem in which the two-dimensional annular domain is replaced by a three-dimensional tubular domain. Though our proofs in Sections 2, 3 and 5 are stated for the case of the symmetric domains above, and particular boundary conditions, our arguments are quite general and do not depend on the symmetry inherent in these samples. In particular, the two dimensional example extends with little difficulty to any domain lying between two star-shaped regions (with a common star centre lying in the inner region) and to a wide class of displacement boundary conditions.

In Section 4, we treat the special example of rotationally symmetric maps of the annulus A. In this setting, more explicit information is obtained on the properties of the equilibria. In particular, it is possible to prove the existence of multiple equilibria by directly seeking radial symmetric minimisers. Finally, in Section 6 we discuss some of the difficulties in extending the results of Sections 2, 3 and 5 to mixed displacement traction problems.

Preliminaries
Given $n, p \in \mathbb{N}$, we denote n-dimensional Lebesgue and p-dimensional Hausdorff measure by $\mathscr{L}^n$, $\mathscr{H}^p$.

Let $\Omega \subset \mathbb{R}^n$ be a domain. We denote by $W^{1,p}(\Omega:\mathbb{R}^n)$ the Sobolev space of maps $\boldsymbol{u}:\Omega \to \mathbb{R}^n$, $\boldsymbol{u} \in L^p(\Omega)$, whose first-order weak derivatives $\nabla\boldsymbol{u}$ exist and satisfy $\nabla\boldsymbol{u} \in L^p(\Omega)$. We will often abbreviate $W^{1,p}(\Omega:\mathbb{R}^n)$ to $W^{1,p}(\Omega)$.

Given $\boldsymbol{u} \in W^{1,p}(\Omega)$, $p \geqq 1$, when dealing with pointwise properties of $\boldsymbol{u}$, or its restriction to lower-dimensional submanifolds, *we will not identify maps that are equal almost everywhere.* In our context, it is advantageous to work with a particular representative of $\boldsymbol{u}$. In particular, the *precise representative* of $\boldsymbol{u}$, denoted $\boldsymbol{u}^*$, is defined for $\boldsymbol{x} \in \Omega$ by

$$\boldsymbol{u}^*(\boldsymbol{x}) = \begin{cases} \lim\limits_{r\to 0} \dfrac{\displaystyle\int_{B_r(x)\cap\Omega} \boldsymbol{u}(y)\,dy}{\displaystyle\int_{B_r(x)\cap\Omega} 1\,dy} & \text{if this limit exists,} \\ \mathbf{0} & \text{otherwise,} \end{cases} \tag{1.4}$$

where $B_r(x) = \{y : |x-y| < r\}$. We refer to [**8**, **13**], and the references therein for important properties of the precise representative.

2. The winding number

The winding number for closed curves in the plane

The idea behind the development given in this subsection is standard in the context of degree theory (see e.g. [**15**]); the precise estimates derived in the proofs will be required later in the paper.

DEFINITION 2.1. Let

$$\gamma:[a,b]\to\mathbb{R}^2, \quad \gamma(r) = \begin{pmatrix} x(r) \\ y(r) \end{pmatrix},$$

be a C^1 curve satisfying $\gamma(a) = \gamma(b)$ (i.e. γ is closed) and $|\gamma(r)| = (x^2(r) + y^2(r))^{\frac{1}{2}} > 0$ for all $r \in [a, b]$. Then the winding number of γ (around the origin) is defined by

$$\text{wind\#}\,\gamma = \frac{1}{2\pi}\int_a^b \frac{x(r)\dfrac{dy(r)}{dr} - y(r)\dfrac{dx(r)}{dr}}{x^2(r)+y^2(r)}\,dr. \tag{2.1}$$

Note that wind# $\gamma \in \mathbb{Z}$ (since e.g. (2.1) can be regarded as the real part of the complex contour integral $1/2\pi i \int_{\tilde{\gamma}} dz$ where $\tilde{\gamma}(r) = x(r) + iy(r)$).

REMARK 2.2. It follows from the above definition and comment that the winding number is continuous in the C^1 topology: i.e. if γ satisfies the hypotheses of the definition, then there exists $\varepsilon > 0$ such that if $\tilde{\gamma}$ is any closed C^1 curve with $\|\gamma - \tilde{\gamma}\|_{C^1([a,b])} < \varepsilon$ then wind# γ = wind# $\tilde{\gamma}$.

We next extend the notion of winding number to closed *continuous* curves.

DEFINITION 2.3. Let $\gamma:[a, b] \to \mathbb{R}^2$ be a continuous curve satisfying $\gamma(a) = \gamma(b)$ and $|\gamma(r)| > 0$ for all $r \in [a, b]$. Let (γ_n), $\gamma_n:[a, b] \to \mathbb{R}^2$, $n \in \mathbb{N}$, be a sequence of closed C^1 curves converging uniformly to γ on $[a, b]$. Then we define wind# $\gamma = \lim_{n\to\infty}$ wind# γ_n.

REMARK 2.4. To see that this extension of the winding number is well defined, suppose that γ is a closed continuous curve with $|\gamma(r)| > 0$ for $r \in [a, b]$. Then there exists $\varepsilon_0 > 0$ such that $|\gamma(r)| > \varepsilon_0 > 0$ for all $r \in [a, b]$. Now suppose that $\gamma_0, \gamma_1:[a, b] \to \mathbb{R}^2$ are closed C^1 curves satisfying $\|\gamma - \gamma_i\|_{C([a,b])} < \varepsilon_0/4$, $i = 0, 1$, and define $\gamma_t:[a, b] \to \mathbb{R}^2$ by $\gamma_t(r) = (1 - t)\gamma_0(r) + t\gamma_1(r)$ for $r \in [a, b]$ for each $r \in [0, 1]$. Then

$$|\gamma_t(r)| = |\gamma_0(r)| - t|\gamma_0(r) - \gamma_1(r)| \geqq \varepsilon_0 - \frac{\varepsilon_0}{4} - \frac{2\varepsilon_0}{4} = \frac{\varepsilon_0}{4} > 0$$

for $r \in [a, b]$ for each $t \in [0, 1]$ and wind# γ_t is well defined for each $t \in [0, 1]$. We now claim that wind# γ_0 = wind# γ_1. To see this, let

$$t^* = \sup\{t \in [0, 1]: \text{wind\# } \gamma_t = \text{wind\# } \gamma_0\}$$

and suppose for a contradiction that $t^* < 1$. Then by the continuity of the winding number in the C^1 topology (see Remark 2.2), the continuity of the map $t \to \gamma_t$ (in the C^1 topology on curves) and the fact that winding number is integer valued, it follows that there exists $\delta > 0$ such that wind# γ_t = wind# γ_0 for $t \in [t^*, t^* + \delta)$, a contradiction. Hence wind# γ_0 = wind# γ_1 as required.

Thus, returning to Definition 2.3, for sufficiently large n, $n \geqq N$ say, $\|\gamma - \gamma_n\|_{C([a,b])} < \varepsilon_0/4$ and hence by the above argument wind# γ_n is constant for all $n \geqq N$. This shows that the winding number is well defined.

From the above argument, we obtain the continuity of the winding number in the uniform topology.

LEMMA 2.5. *Let* $\gamma:[a, b] \to \mathbb{R}^2$ *be a closed continuous curve. Let* $|\gamma(t)| > \varepsilon_0 > 0$ *for all* $t \in [a, b]$. *Let* $\tilde{\gamma}:[a, b] \to \mathbb{R}^2$ *be a closed continuous curve satisfying* $\|\gamma - \tilde{\gamma}\|_{C([a,b])} < \varepsilon_0/4$; *then* wind# $\tilde{\gamma}$ = wind# γ. (*Hence the winding number is continuous in the uniform topology.*)

The winding number for maps of the annulus

Let $0 < a < b$ and let $A = \{x \in \mathbb{R}^2 : a < |x| < b\}$. Let the class of admissible deformations of A be defined by

$$\mathscr{A} = \{u \in W^{1,1}(A) : u : A \to \bar{A} \text{ almost everywhere, } u = \text{identity on } \partial A\}. \quad (2.2)$$

The condition $u: A \to \bar{A}$ almost everywhere denotes the existence of a set $N \subset A$, $\mathscr{L}^1(N) = 0$, such that $u(x) \in \bar{A}$ for all $x \in A \backslash N$. We will demonstrate that the results presented thus far in Section 2 can be used to identify homotopy classes of maps in

$\mathscr{A}$. Roughly speaking, the idea is the following: given $\boldsymbol{u} \in \mathscr{A}$, we transform to polar coordinates and write $\boldsymbol{u} = \boldsymbol{u}(r, \theta)$, and it then follows that $\boldsymbol{u}(\cdot, \theta)$ is absolutely continuous (i.e. in $W^{1,1}((a, b))$ for almost every $\theta \in (0, 2\pi)$. Hence, by the embedding of $W^{1,1}((a, b))$ in $C([a, b])$, we may assume that

$$\gamma_\theta(r) = \frac{\boldsymbol{u}(r, \theta)}{|\boldsymbol{u}(r, \theta)|} \tag{2.3}$$

is a closed continuous curve in $\mathbb{R}^2$ for almost every $\theta \in [0, 2\pi)$. Thus we may define wind# γ_θ for almost every $\theta \in [0, 2\pi)$. To adopt this approach in a way which is preserved under weak convergence, we will require mapping properties of the precise representatives (see (1.4)) of maps in $\mathscr{A}$ on radial lines, and their properties under weak convergence. These are given in the following lemmas.

LEMMA 2.6. *Let $\boldsymbol{u} \in \mathscr{A}$; then there exists $N \subset [0, 2\pi)$, $\mathscr{L}^1(N) = 0$, such that for each $\theta \in [0, 2\pi)\backslash N$, $\boldsymbol{u}^*(r, \theta)$ is an absolutely continuous curve lying in $\bar{A}$ with*

$$\lim_{r \to b^-} \boldsymbol{u}^*(r, \theta) = b(\cos\theta, \sin\theta), \quad \lim_{r \to a^+} \boldsymbol{u}^*(r, \theta) = a(\cos\theta, \sin\theta).$$

Proof. We work in polar coordinates (r, θ). Let $\boldsymbol{u}^*(r, \theta)$, $(r, \theta) \in P = [a, b] \times [0, 2\pi)$ be the precise representative of $\boldsymbol{u}$. Then $\boldsymbol{u} = \boldsymbol{u}^*$ $\mathscr{L}^2$-almost everywhere. Thus there exists $Q \subset P$, $\mathscr{L}^2(Q) = 0$, such that

$$\boldsymbol{u}^*(r, \theta) \in \bar{A} \quad \text{for all } (r, \theta) \in P\backslash Q. \tag{2.4}$$

By Fubini's Theorem, there exists $M \subset [0, 2\pi)$, $\mathscr{L}^1(M) = 0$, such that $Q_\theta = \{r \in (a, b) : (r, \theta) \in Q\}$ has $\mathscr{L}^1$ measure zero for each $\theta \in [0, 2\pi)\backslash M$.

Now, by the absolute continuity properties of the precise representative $\boldsymbol{u}^*$ (see [**13**, Proposition 2.8]), there exists $\tilde{M} \subset [0, 2\pi)$, $\mathscr{L}^1(\tilde{M}) = 0$, such that $\boldsymbol{u}^*(\cdot, \theta)$ is absolutely continuous on (a, b) for each $\theta \in [0, 2\pi)\backslash\tilde{M}$. Now let $N = M \cup \tilde{M}$. We claim that $\{\boldsymbol{u}^*(r, \theta) : r \in (a, b)\} \subset \bar{A}$ for each $\theta \in [0, 2\pi)\backslash N$. (Otherwise, if $\boldsymbol{u}^*(\tilde{r}, \theta) \notin \bar{A}$ for some $\tilde{r} \in (a, b)$, $\theta \in [0, 2\pi)\backslash N$, then, by continuity of $\boldsymbol{u}^*(\cdot, \theta)$, we obtain a contradiction of (2.4) and the definition of N.

Moreover, by [**12**, Lemma 3.1.1], we may assume without loss of generality that $\lim_{r \to b^-, a^+} \boldsymbol{u}^*(r, \theta)$ exist for each $\theta \in [0, 2\pi)\backslash N$ and that by [**8**, Section 4.3] that these limits equal $b(\cos\theta, \sin\theta)$, $a(\cos\theta, \sin\theta)$, respectively.

Hence we may assume by the last lemma that

$$\gamma_\theta(r) = \frac{\mathbf{u}^*(r, \theta)}{|\boldsymbol{u}^*(r, \theta)|} \quad r \in [a, b]$$

is a closed continuous curve with a well-defined winding number (around the origin) for each $\theta \in [0, 2\pi)\backslash N$. □

The next lemma demonstrates that $\mathscr{A}$ is sequentially weakly closed.

LEMMA 2.7. *Let $(\boldsymbol{u}_n) \subset \mathscr{A}$ converge weakly in $W^{1,p}(A)$, $p \geqq 1$, to $\boldsymbol{u} \in W^{1,p}(A)$. Then $\boldsymbol{u} \in \mathscr{A}$.*

Proof. It follows from the boundedness of the trace operator that $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{x}$ for $\boldsymbol{x} \in \partial A$ (in the sense of trace). By Lemma 2.6, there exists a set $N \subset [0, 2\pi)$, $\mathscr{L}^1(N) = 0$, such that each $\boldsymbol{u}_n$, $n \in \mathbb{N}$, satisfies the mapping properties given in Lemma 2.6 for each $\theta \in [0, 2\pi)\backslash N$. Now by the results of [**13**, Lemma 2.9] there exists a set $S \subset [0, 2\pi)$,

$\mathscr{L}^1(S)=0$, such that for each $\theta\in[0,2\pi)\backslash S$ there exists a subsequence $(u_{n_k})_{k=1}^{\infty}$ satisfying $u^*_{n_k}(\cdot,\theta)\rightharpoonup u^*(\cdot,\theta)$ as $k\to\infty$ in $W^{1,p}((a,b))$. Let $\tilde{S}=S\cup N$. Then, since the embedding of $W^{1,p}((a,b))$ in $C([a,b])$ maps weakly convergent sequences to strongly convergent ones, it follows that $u^*_{n_k}(\cdot,\theta)\to u^*(\cdot,\theta)$ in $C([a,b])$ as $k\to\infty$ for each $\theta\in[0,2\pi)\backslash\tilde{S}$. Hence $\{u^*(r,\theta):r\in(a,b)\}\subset\bar{A}$ for each $\theta\in[0,2\pi)\backslash\tilde{S}$ (since $\{u^*_{n_k}(r,\theta):r\in(a,b)\}\subset\bar{A}$ for each $\theta\in[0,2\pi)\backslash\tilde{S}$). Thus $u^*:A\to\bar{A}$ almost everywhere and hence $u:A\to\bar{A}$ almost everywhere as required. □

Definition 2.8. For each $N\in\mathbb{Z}$, we say that $u\in\mathscr{A}$ satisfies the homotopy condition (H_N) if

$$\mathrm{wind}\#\,\gamma_\theta=N\quad\text{for almost every }\theta\in[0,2\pi),\tag{2.5}$$

where γ_θ is defined in terms of u by (2.3).

The next result uses the last two lemmas and shows that the condition (H_N) on the winding number is preserved under sequential weak convergence.

Lemma 2.9. *Let $N\in\mathbb{N}$ and let*

$$\mathscr{A}^N=\{u\in\mathscr{A}:u\text{ satisfies }(H_N)\}.\tag{2.6}$$

If $(u_n)\subset\mathscr{A}^N$ satisfies $u_n\rightharpoonup u$ in $W^{1,p}(A)$, $p\geqq 1$, then $u\in\mathscr{A}^N$.

Proof. By the results of Lemma 2.6, we may assume the existence of a set $\tilde{N}\subset[0,2\pi)$, $\mathscr{L}^1(\tilde{N})=0$, such that each u_n, $n\in\mathbb{N}$ and u satisfy the mapping, absolute continuity and limiting properties of Lemma 2.6 for each $\theta\in[0,2\pi)\backslash\tilde{N}$. Now, by the arguments of Lemma 2.7, we may suppose without loss of generality that $\tilde{N}\subset\tilde{S}\subset[0,2\pi)$, $\mathscr{L}^1(\tilde{S})=0$, and that for each $\theta\in[0,2\pi)\backslash\tilde{S}$ there is a subsequence (u_{n_k}) such that $u^*_{n_k}(\cdot,\theta)\to u^*(\cdot,\theta)$ in $C([a,b])$. Hence, by Lemma 2.5, since each u_{n_k} satisfies (H_N), it follows that u also satisfies (H_N).

Remark 2.10. Note that if $u\in\mathscr{A}\cap C(\bar{A})$ then, by Lemma 2.5, condition (2.5) holds for all $\theta\in[0,2\pi)$.

3. Existence of multiple equilibria in two dimensions

We assume throughout this section that the stored energy function $W:M^{2\times 2}\to\bar{\mathscr{R}}$ satisfies the following hypotheses:

- (H1) W is continuous and $W(F)=\infty$ if and only if $\det F\leqq 0$;
- (H2) W is polyconvex, i.e. $W(F)=g(F,\det F)$ for all $F\in M^{2\times 2}_+$, where $g:M^{2\times 2}\times(0,\infty)\to\mathbb{R}$ is convex;
- (H3) $W(F)\geqq C_1|F|^p+C_2$ for all $F\in M^{2\times 2}$, $d\in(0,\infty)$, for some $P\geqq 2$, where $C_1>0$ and C_2 are constants.

Theorem 3.1. *Let W satisfy* (H1), (H2), (H3); *Then E attains a minimum on*

$$\mathscr{A}^N=\{u\in\mathscr{A}:u\text{ satisfies }(H_N)\}.$$

Proof. The existence of a minimiser of E on

$$\tilde{\mathscr{A}}=\{u\in W^{1,P}(A):u(x)=x\text{ for }x\in\partial A,\ E(u)<+\infty,\ \det\nabla u>0\}\text{ a.e.}\tag{3.1}$$

follows immediately from [4, Theorem 6.1 and the comments following the proof].

(The fact that $\tilde{\mathscr{A}}$ is nonempty follows, for example, by considering a map of the form (4.1) with $\rho(R) \equiv R$ and $\psi(R) = 2N\pi(R-a)/(b-a)$.) The proof of our theorem follows from observing that if $(u_n) \subset \mathscr{A}^N$ is a minimising sequence for E on $\mathscr{A}^N$, then by the arguments of [**4**] there is a weakly convergent subsequence $u_{n_k} \rightharpoonup u$ in $W^{1,P}(A)$ as $k \to \infty$ for some $u \in \tilde{\mathscr{A}}$ and $E(u) \leqq \liminf_{k\to\infty} E(u_{n_k})$ and u satisfies (H_N) by Lemma 2.9. Hence $u \in \mathscr{A}^N$ is a minimiser of E on $\mathscr{A}^N$. □

REMARK 3.2. If W satisfies (H1), (H2), (H3) with $P > 2$, it follows from Theorem 3.1 that E has infinitely many strong local minimisers: fix $N \in \mathbb{N}$ and let u^N be a minimiser of E on $\mathscr{A}^N$. Since $W^{1,P}(A) \subset C(\bar{A})$, if $\tilde{u} \in \mathscr{A}$ satisfies $\|u^N - \tilde{u}\|_{C(\bar{A})} < a/8$ then, by Lemma 2.5, wind# $\tilde{u}(\cdot, \theta) = N$ for all $\theta \in [0, 2\pi)$ and so $\tilde{u} \in \mathscr{A}^N$. Hence $E(u^N) \leqq E(\tilde{u})$ and so u^N is a strong local minimiser of E on $\mathscr{A}$.

REMARK 3.3. Note that if $P > 2$ then, by the compact Sobolev embedding, we may assume further in the proof of Theorem 3.1 that $u_{n_k} \to u$ in $C(\bar{\Omega})$. Hence the fact that u satisfies (H_N) if the minimising sequence (u_n) satisfies (H_N) is immediate by Lemma 2.5 without recourse to the results of Lemma 2.9.

REMARK 3.4. Since W is polyconvex by assumption, standard arguments (see e.g. [**1**]) imply that W is quasiconvex and in particular that, for $p \geqq 2$,

$$E(u) \geqq E(u^{hom}) \quad \text{for all } u \in \mathscr{A},$$

where $u^{\text{hom}}(x) \equiv x$. Thus this homogeneous map is the global energy minimiser (with no winding number constraint).

REMARK 3.5. The following straightforward argument demonstrates that, for frame indifferent isotropic stored energy functions W, if u is a strict local minimiser of E in $\mathscr{A}^N$, then u is rotationally symmetric. Let u be such a strong local minimiser then $u \in C(\bar{\Omega})$ by [**19**, Theorem 5], let

$$Q(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \quad \varphi \in \mathbb{R}$$

and consider $u_\varphi(x) = Q^T(\varphi)u(Q(\varphi)x)\quad x \in \bar{A}$. Then $u_\varphi|_{\partial A} = u|_{\partial A}$, $\|u - u_\varphi\|_{C(\bar{\Omega})} \to 0$ as $\varphi \to 0$ and

$$E(u_\varphi) = \int_A W(Q^T \nabla u(Qx)Q)\, dx = \int_A W(\nabla u(x))\, dx.$$

Since, by assumption, u is a strict local minimiser, it follows that $u_\varphi \equiv u$ for all φ sufficiently small. A straightforward continuity argument now gives that $u_\phi \equiv u$ for all φ and so u is rotationally symmetric.

Finally in this section, we give a simple lower bound for $\inf_{\mathscr{A}^N} E$ in terms of the winding number N.

LEMMA 3.6. *Let W satisfy* (H3); *then*

$$\inf_{\mathscr{A}^N} E \geqq (\text{mes } A)^{1-P} \left(\sqrt{2}\pi \frac{a^2}{b} N \right)^P.$$

Proof. First, observe that by Hölder's inequality,

$$\int_A |\nabla u|^p \geqq (\text{mes } A)^{1-p} \|\nabla u\|_{L^1}^p \tag{3.2}$$

and that if $u \in \mathscr{A}^N$, then

$$\begin{aligned}\|\nabla u\|_{L^1} &\geqq \frac{1}{\sqrt{2}} \int_0^{2\pi} \int_a^b R \left[\left| \frac{\partial u^1}{\partial R} \right| + \frac{1}{R} \left| \frac{\partial u^1}{\partial \theta} \right| + \left| \frac{\partial u^2}{\partial R} \right| + \frac{1}{R} \left| \frac{\partial u^2}{\partial \theta} \right| \right] dR\, d\theta \\ &\geqq \frac{a^2}{\sqrt{2}b} \int_0^{2\pi} \int_a^b R \frac{\left[\left| u^2 \frac{\partial u^1}{\partial R} \right| + \left| u^1 \frac{\partial u^2}{\partial R} \right| \right]}{|u|^2} dR\, d\theta \\ &\geqq \frac{a^2}{\sqrt{2}b} \int_0^{2\pi} \int_a^b \frac{u^2 \frac{\partial u^1}{\partial R} - u^1 \frac{\partial u^2}{\partial R}}{|u|^2} dR\, d\theta \\ &= \sqrt{2}\pi \frac{a^2}{b} N, \end{aligned} \tag{3.3}$$

by (2.1). Hence by (H3), (3.2) and (3.3) it follows that

$$E(u) \geqq (\text{mes } A)^{1-p} \left(\sqrt{2}\pi \frac{a^2}{b} N \right)^p. \quad \square$$

Remark 3.7. The results of [**2**] (see also [**5**, **13**]) give conditions under which a minimiser u_0 of Theorem 3.1 (for some $N \in \mathbb{N}$) is a weak solution of a corresponding system of Euler–Lagrange equations. The basic idea is to consider two types of variations which preserve the invertibility condition (1.2): 'inner' variations, which are of the form:

(i) $$u_\varepsilon(x) = u_0(\Phi_\varepsilon(x)) \quad \text{where } \Phi_\varepsilon(x) = x + \varepsilon \Psi(x) \text{ with } \Psi \in C_0^1(A), \tag{3.4}$$

and 'outer' variations which are of the form:

(ii) $$u_\varepsilon(x) = \Phi_\varepsilon(u_0(x)) \quad \text{where } \Phi_\varepsilon(y) = y + \varepsilon \Psi(y) \text{ with } \Psi \in C_0^1(u_0(A)), \tag{3.5}$$

(In the above, u_0 is a minimiser, $\varepsilon \in \mathbb{R}$ is a parameter, and the Euler–Lagrange equations are derived by setting $(d/d\varepsilon)E(u_\varepsilon) = 0$.)

Notice that by Lemma 2.5, both inner and outer variations preserve the condition (H_N) if $\varepsilon \in \mathbb{R}$ is sufficiently small.

In case (i), the corresponding Euler–Lagrange equations are

$$\frac{\partial}{\partial x^\alpha} \left[W(\nabla u(x)) \delta_\alpha^i - u_i^\beta(x) \frac{\partial W}{\partial F_\alpha^\beta} (\nabla u(x)) \right] = 0 \quad \text{in } \mathscr{D}'(A) \quad i = 1, 2. \tag{3.6}$$

(The expression in square brackets is sometimes referred to as the Energy-Momentum Tensor.)

In case (ii), the corresponding Euler–Lagrange equations are

$$\frac{\partial}{\partial u^\alpha} (T_{i\alpha}(u)) = 0 \quad \text{in } \mathscr{D}'(u_0(A)) \quad i = 1, 2, \tag{3.7}$$

where $T = (T_{i\alpha})$ is the Cauchy stress tensor defined on $\boldsymbol{u}_0(A)$ by

$$T(\boldsymbol{u}_0(\boldsymbol{x})) = \frac{1}{\det \nabla \boldsymbol{u}_0(\boldsymbol{x})} \frac{\partial W}{\partial F}(\nabla \boldsymbol{u}_0(\boldsymbol{x}))(\nabla \boldsymbol{u}_0(\boldsymbol{x}))^T. \tag{3.8}$$

It is interesting to note that the arguments leading to the proof of Theorem 3.1 can also be used to proves the existence of minimisers of functionals such as

$$\tilde{E}(\boldsymbol{u}) = \int_A |\nabla \boldsymbol{u}(\boldsymbol{x})|^p \, dx, \quad p \geqq 2,$$

on sets of the form

$$\bar{\mathscr{A}}_N = \{\boldsymbol{u} \in W^{1,p}(A) : \det \nabla \boldsymbol{u} \geqq 0 \text{ almost everywhere}, \boldsymbol{u}|_{\partial A} = \text{identity}, \boldsymbol{u} \text{ satisfies } (H_N)\}$$

for each $N \in \mathbb{N}$. (Note that equality is allowed in the determinant constraint defining $\bar{\mathscr{A}}_N$.)

Since $\tilde{E}$ is strictly convex, the unique smooth invertible solution of (3.6) and (3.7) satisfying the boundary conditions is $\boldsymbol{u}_0(\boldsymbol{x}) \equiv \boldsymbol{x}$. Hence for each $N \geqq 1$, it follows that a corresponding minimiser $\boldsymbol{u}_0$ of $\tilde{E}$ on $\bar{\mathscr{A}}_N$ cannot be smooth and invertible. This can occur, for example, when $\det \nabla \boldsymbol{u}_0 = 0$ on a set of nonzero measure. In such cases, the arguments in [**2, 5, 13**], leading to the system (3.7) break down (as (3.8) is not defined), but those leading to (3.6) are still valid for these minimisers.

4. An example

Rotationally symmetric maps of the annulus

EXAMPLE 4.1. In this section, we consider maps $\boldsymbol{u} : \bar{A} \to \bar{A}$, $A = \{\boldsymbol{x} \in \mathbb{R}^2 : a < |\boldsymbol{x}| < b\}$, which are of the form

$$\boldsymbol{u}(\boldsymbol{x}) = \rho(R) \begin{pmatrix} \cos(\theta + \psi(R)) \\ \sin(\theta + \psi(R)) \end{pmatrix} \quad \text{for all } \boldsymbol{x} \in \bar{A}, \tag{4.1}$$

where (R, θ) are polar coordinates in the plane, $R = |\boldsymbol{x}|$, and $\rho : [a, b] \to [a, b]$ $\psi : [a, b] \to \mathbb{R}$. It follows that, for sufficiently smooth deformations of this type,

$$\nabla \boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} \rho'(R) \cos(\theta + \psi) - \rho(R)\psi' \sin(\theta + \psi) \\ \rho'(R) \sin(\theta + \psi) + \rho(R)\psi' \cos(\theta + \psi) \end{pmatrix} \otimes \frac{\boldsymbol{x}}{R}$$

$$+ \begin{pmatrix} -\dfrac{\rho}{R} \sin(\theta + \psi) \\ \\ \dfrac{\rho}{R} \cos(\theta + \psi) \end{pmatrix} \otimes \frac{1}{R} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}. \tag{4.2}$$

A straightforward calculation shows that

$$|\nabla \boldsymbol{u}|^2 = \text{tr}\,[\nabla \boldsymbol{u}(\nabla \boldsymbol{u})^T] = \left(\frac{\rho}{R}\right)^2 + (\rho')^2 + (\rho\psi')^2 \tag{4.3}$$

and (using the identity $\det(\boldsymbol{a}\otimes\boldsymbol{a}+\boldsymbol{b}\otimes\boldsymbol{b}) = (a_1 b_2 - a_2 b_1)^2$) that

$$(\det \nabla\boldsymbol{u})^2 = \det[\nabla\boldsymbol{u}\nabla\boldsymbol{u}^T] = \left[\rho'\frac{\rho}{R}\right]^2. \tag{4.4}$$

The next lemma is straightforward to verify.

LEMMA 4.2. *If $(\rho, \psi) \in W^{1,1}((a, b))$, $\rho:[a,b]\to(0,\infty)$, $\psi:[a,b]\to\mathbb{R}$ and satisfy*

$$\int_a^b R\left[\left(\frac{\rho}{R}\right)^P + (\rho')^P + (\rho\psi')^P\right] dR < +\infty, \tag{4.5}$$

for some $P \geqq 1$, then the corresponding map $\boldsymbol{u}$ given by (4.1) lies in $W^{1,P}(A)$.

For simplicity, we restrict attention to a class of polyconvex stored energy functions of the form

$$W(F) = \frac{1}{2}|F|^2 + h(\det F) \quad \text{for } F \in M^{2\times 2}, \tag{4.6}$$

where $h:(0,\infty)\to\mathbb{R}^+$ is convex, C^2 and satisfies

$$h(d)\to\infty \quad \text{as } d\to 0, \infty. \tag{4.7}$$

We extend h to be defined on $\mathbb{R}$ by setting $h(d) = +\infty$ if $d \in (-\infty, 0]$. Notice that W is both frame indifferent and isotropic; i.e. that

$$W(F) = W(QF) = W(FQ) \quad \text{for all } F \in M_+^{2\times 2}, \quad \text{for all } Q \in SO(2)$$

(see e.g. [**6**, **14**, **20**]). For stored energy functions of this form, it follows from (4.3), (4.4) that the total stored energy (1.3) corresponds to a rotationally symmetric deformation of the form (4.1) is given by

$$E(\boldsymbol{u}) = 2\pi I(\rho,\psi) = 2\pi\int_a^b R\left[\frac{1}{2}\left((\rho')^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\psi')^2\right) + h\left(\rho'\frac{\rho}{R}\right)\right] dR. \tag{4.8}$$

For each $N \in \mathbb{N}\cup\{0\}$, we seek equilibria by minimising I on the set

$$\mathscr{A}_N^{\text{sym}} = \{(\rho,\psi)\in W^{1,1}((a,b)) : \rho(a) = a,\ \rho(b) = b,\ \rho'(R) > 0 \text{ a.e. on } (a,b),\ \psi(a) = 0,\ \psi(b) = 2N\pi\}. \tag{4.9}$$

The condition $\rho' > 0$ ensures that the maps (4.1) satisfy (1.2) and the condition $\psi(b) = 2N\pi$ corresponds to the homotopy condition (H_N) in this symmetric setting. Let $(\rho_n, \psi_n) \in \mathscr{A}_N^{\text{sym}}$ be a minimising sequence for I on $\mathscr{A}_N^{\text{sym}}$, i.e. $I(\rho_n, \psi_n) \to \inf_{\mathscr{A}_N^{\text{sym}}} I$ as $n\to\infty$; then

$$I(\rho_n, \psi_n) \geqq a\int_a^b \left[(\rho_n')^2 + \frac{1}{b^2}(\rho_n)^2 + a^2(\psi_n')^2\right] dR \quad \forall n \in \mathbb{N}.$$

Hence $((\rho_n, \psi_n))_{n=1}^\infty$ is a bounded sequence in $W^{1,2}((a,b))$ and hence has a weakly convergent subsequence, still labelled $((\rho_n, \psi_n))$, converging weakly to some $(\rho,\psi)\in W^{1,2}((a,b))$ satisfying $\rho(a) = a$, $\rho(b) = b$, $\psi(a) = 0$, $\psi(b) = 2N\pi$. Standard results imply that I is sequentially weakly lower semicontinuous on such a subsequence and it then follows that

$$\inf_{\mathscr{A}_N^{\text{sym}}} I = \liminf_{n\to\infty} I(\rho_n, \psi_n) \geqq I(\rho, \psi)$$

so that, in particular $I(\rho, \psi) < \infty$. Thus, by the properties of h, it follows that $\rho'(R) > 0$ a.e. $R \in (a, b)$ and so $(\rho, \psi) \in \mathscr{A}_N^{\text{sym}}$. Hence

$$I(\rho, \psi) = \inf_{\mathscr{A}_N^{\text{sym}}} I.$$

Standard regularity arguments then show that $(\rho, \psi) \in C^2((a, b))$ and satisfy the Euler–Lagrange equations:

$$\text{(i)} \qquad \frac{d}{dR}\left[R\rho'(R) + \rho h'\left(\rho' \frac{\rho}{R}\right)\right] = \frac{\rho}{R} + R\rho(\psi')^2 + \rho' h'\left(\rho' \frac{\rho}{R}\right);$$

$$\text{(ii)} \qquad \frac{d}{dR}\left[R\rho^2(R)\psi'(R)\right] = 0, \quad R \in (a, b).$$

REMARK 4.3. A straightforward, though slightly lengthy, calculation shows that any smooth solution of (i) and (ii) above gives rise to a corresponding solution $\boldsymbol{u}$ (by (4.1)) of the full Euler–Lagrange equations for E (given by (1.3)), i.e.

$$\frac{\partial}{\partial x^\alpha}\left[\frac{\partial W}{\partial F^i_\alpha}(\nabla \boldsymbol{u}(\boldsymbol{x}))\right] = 0 \quad \boldsymbol{x} \in \Omega = A, \quad i = 1, 2,$$

for energy functions of the form (4.6).

REMARK 4.4. Integration of (ii) shows that if $(\rho, \psi) \in \mathscr{A}_N^{\text{sym}}$ is a minimiser of I on $\mathscr{A}_N^{\text{sym}}$, then $R\rho^2(R)\psi'(R) = c = \text{constant}$ for $R \in (a, b)$. Hence, if $c \neq 0$, then ψ' is one-signed. The case $c = 0$ can only occur if $N = 0$ (by the boundary conditions) and in this case $\psi \equiv 0$. Thus for any minimiser (ρ, ψ) we have that the corresponding angle of twist ψ is monotonic in R, i.e. the corresponding map (4.1) of the annulus is a monotone twist map.

REMARK 4.5. The following example shows that the homotopy condition (H_N) is not preserved in general under convergence in L^P, $1 \leqq P < \infty$. We consider a sequence of radial maps $(\boldsymbol{u}_k)$ of the form (4.1) with corresponding functions (ρ_k, ψ_k), $k \in \mathbb{N}$, where

$$\rho_k(R) \equiv R \quad \text{for all } k$$

and

$$\psi_k(R) = \begin{cases} 0 & \text{for } R \in [a, b - 1/k] \\ 2\pi[k(R - b) + 1] & \text{for } R \in [b - 1/k, b]. \end{cases}$$

Then it is easily verified that $(\boldsymbol{u}_k)$ is bounded in $W^{1,1}(A)$ and that $\boldsymbol{u}_k \to \boldsymbol{u}$, as $k \to \infty$ in L^P for all $1 \leqq P < \infty$, where $\boldsymbol{u}$ is the identity map. However, $\boldsymbol{u}_k$ satisfies (H_1) for all k but $\boldsymbol{u}$ satisfies (H_0).

The example of this section can be extended to a much wider class of isotropic, frame-indifferent, stored energy functions. It follows from our results that for each $N \in \mathbb{N}_0$, I attains a minimum in the class of rotationally symmetric maps $\mathscr{A}_N^{\text{sym}}$. By

Theorem 3.1, we have that E attains a minimum on $\mathscr{A}^N$ (defined by (3.1)) with no assumption of rotational symmetry. It is an open question at present whether, for the stored energy function of the example, any nonhomogeneous minimiser whose existence is guaranteed by Theorem 3.1 is necessarily rotationally symmetric (a possible approach is given in [**16**]). Note that the existence of infinitely many locally minimising critical points of the energy functional does not necessarily imply the existence of infinitely many unstable critical points (as one might expect by a Morse-type argument) since each local minimiser may sit in an infinitely high potential well due to condition (4.7).

5. Existence of multiple equilibria in three dimensions

In this section, we present a three-dimensional example demonstrating the existence of multiple equilibria for a Dirichlet boundary-value problem from nonlinear elasticity.

Let $\Omega \subset \mathbb{R}^3$ be the 'hollow tube' (see Fig. 5.1) of radius c and thickness $(b-a)$:

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : a^2 \leqq [(x^2+y^2)^{\frac{1}{2}} - c]^2 + z^2 \leqq b^2\}, \tag{5.1}$$

where $a, b, c \in \mathbb{R}$ are constants satisfying $0 < a < b < c$; i.e. Ω is the domain lying between the two tori T_a, T_b, where

$$T_r = \{(x, y, z) \in \mathbb{R}^3 : [(x^2+y^2)^{\frac{1}{2}} - c]^2 + z^2 \leqq r^2\}, \quad 0 < r < c.$$

We proceed analogously to the two-dimensional case considered earlier.

The winding numbers for curves on the torus T_b

Let $\gamma : [0, 1] \to T_b$ be the closed C^1 curve. We define two winding numbers for any such curve.

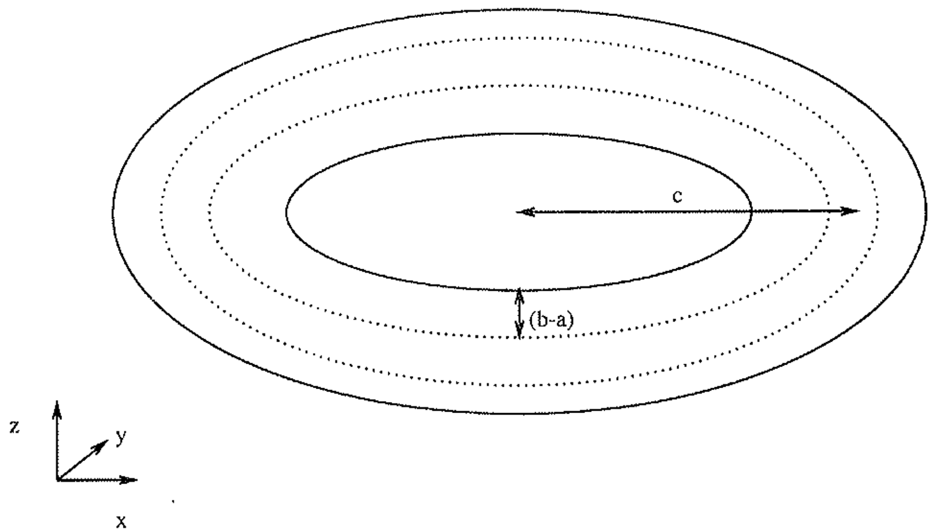

Figure 5.1

DEFINITION 5.1. The *axial* winding number, denoted by $\text{wind}^{(1)}\#\,\gamma$, is defined to be the winding number of the projection of γ onto the x, y-plane, i.e.

$$\text{wind}^{(1)}\#\,\gamma = \frac{1}{2\pi}\int_0^1 \frac{x(t)\dot{y}(t) - y(t)\dot{x}(t)}{x^2(t) + y^2(t)}\,dt,$$

where $\gamma(t) = (x(t), y(t), z(t))$ for $t \in [0, 1]$.

We next define a 'tubular' winding number for which we require the following axially symmetric, curl-free, unit tangent vector field $\boldsymbol{t}$ on the torus T_b:

$$\boldsymbol{t}(\boldsymbol{x}) = \frac{1}{b}\begin{pmatrix} \dfrac{-zx}{(x^2+y^2)^{\frac{1}{2}}} \\ \dfrac{-zy}{(x^2+y^2)^{\frac{1}{2}}} \\ (x^2+y^2)^{\frac{1}{2}} - c \end{pmatrix} \quad \text{for all } \boldsymbol{x} = (x, y, z) \in T_b. \tag{5.2}$$

Notice that, locally, $\boldsymbol{t}(\boldsymbol{x}) = \nabla\psi(\boldsymbol{x})$, where

$$\psi(\boldsymbol{x}) = b\tan^{-1}\left(\frac{z}{(x^2+y^2)^{\frac{1}{2}} - c}\right) \quad \text{for } (x^2+y^2) \neq c^2.$$

DEFINITION 5.2. The *tubular* winding number of γ, denoted $\text{wind}^{(2)}\#\,\gamma$, is given by

$$\text{wind}^{(2)}\#\,\gamma = \frac{1}{2\pi b}\int_0^1 \boldsymbol{t}(\gamma(t))\cdot\dot{\gamma}(t)\,dt. \tag{5.3}$$

REMARK 5.3. It is a consequence of Stokes' Theorem that $\text{wind}^{(2)}\#\,\gamma$ is integer-valued. To see this, first observe that if γ_0 and γ_1 are homotopic curves on T_b and $\gamma(s, t)$, $\gamma: [0, 1] \times [0, 1] \to T_b$, is a smooth homotopy ($s \in [0, 1]$ being the homotopy parameter) with $\gamma(0, \cdot) = \gamma_0(\cdot)$, $\gamma(1, \cdot) = \gamma_1(\cdot)$, then $\text{wind}^{(2)}\#\,\gamma_0 = \text{wind}^{(2)}\#\,\gamma_1$ by Stokes' Theorem applied to the surface S 'swept out' by the homotopy, where $S = \{(x, y, z) \in T_b : (x, y, z) = \gamma(s, t) \text{ for some } s \in [0, 1], t \in [0, 1]\}$.

It is a basic result from algebraic topology (see e.g. [**9**]) that the fundamental group of the torus is commutative and that we can homotope any given curve $\tilde{\gamma}: [0, 1] \to T_b$ to a basic curve of the form $\boldsymbol{\lambda}^{(n)} + \boldsymbol{\mu}^{(m)}$, $m, n \in \mathbb{Z}$, where $+$ denotes the join of the curves and $\boldsymbol{\lambda}^{(n)}$, $n \in \mathbb{N}$, denotes the join of n curves $\boldsymbol{\lambda}^{(1)} + \ldots + \boldsymbol{\lambda}^{(1)}$, where

$$\boldsymbol{\lambda}^{(1)}(t) = ([c-b]\cos 2\pi t, [c-b]\sin 2\pi t, 0), \quad t \in [0, 1],$$

and $\boldsymbol{\lambda}^{(-n)}$ for $n \in \mathbb{N}$ denotes the same curve with the reverse orientation. Similarly, $\boldsymbol{\mu}^{(m)}$ denotes the join of m curves $\boldsymbol{\mu}^{(1)} + \ldots + \boldsymbol{\mu}^{(1)}$, where

$$\boldsymbol{\mu}^{(1)}(t) = (c + b\cos 2\pi t, 0, b\sin 2\pi t), \quad t \in [0, 1],$$

(see Fig. 5.2). A straightforward calculation, using the additivity of the definition (5.3) and the fact that $\text{wind}^{(2)}\#\,(\boldsymbol{\lambda}^{(1)}) = 0$, then shows that

$$\text{wind}^{(2)}\#\,(\boldsymbol{\lambda}^{(n)} + \boldsymbol{\mu}^{(m)}) = m \quad \text{for all } n, m \in \mathbb{Z}.$$

Following our earlier development, we next extend the notion of axial and tubular winding number from C^1 to continuous curves γ.

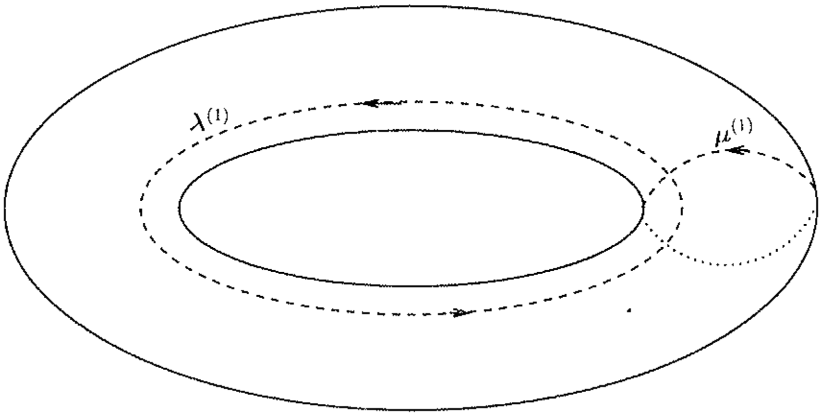

Figure 5.2

DEFINITION 5.4. Let $\gamma:[0,1]\to T_b$ be a continuous curve. Then we define

$$\mathrm{wind}^{(1)}\#\,\gamma=\lim_{n\to\infty}\mathrm{wind}^{(1)}\#\,\gamma_n,$$

$$\mathrm{wind}^{(2)}\#\,\gamma=\lim_{n\to\infty}\mathrm{wind}^{(2)}\#\,\gamma_n,$$

where $\gamma_n:[0,1]\to T_b$, $n\in\mathbb{N}$, is any sequence of C^1 curves converging uniformly to γ.

REMARK 5.5. An analogous argument to that given in Remark 2.4 shows that $\mathrm{wind}^{(i)}\#\,\gamma$, $i=1,2$, is well defined and continuous in the uniform topology.

The winding numbers of maps of the tubular domain Ω

Let Ω be the hollow tube defined by (5.1). Analogously to the two-dimensional case considered in Section 3, we consider maps $\boldsymbol{u}$ in the class

$$\mathscr{A}=\{\boldsymbol{u}\in W^{1,1}(\Omega):\boldsymbol{u}:\Omega\to\bar{\Omega}\text{ a.e. }\boldsymbol{u}|_{\partial\Omega}=\text{identity}\}.$$

In order to define homotopy classes for such maps, we first define the 'tubular projection' operator $\boldsymbol{P}:\Omega\to T_b$ given by

$$\boldsymbol{P}(\boldsymbol{x})=\boldsymbol{P}(x,y,z)=c\left(\frac{x}{\tilde{r}},\frac{y}{\tilde{r}},0\right)+b\,\frac{\left(x\left[1-\frac{c}{\tilde{r}}\right],y\left[1-\frac{c}{\tilde{r}}\right],z\right)}{[(\tilde{r}-c)^2+z^2]^{\frac{1}{2}}},$$

$$\text{for all }\boldsymbol{x}=(x,y,z)\in\Omega,\tag{5.4}$$

where $\tilde{r}=(x^2+y^2)^{\frac{1}{2}}$.

DEFINITION 5.6. Given $M,N\in\mathbb{Z}$, we say that $\boldsymbol{u}$ satisfies the homotopy condition $(H_{M,N})$ if

$$\mathrm{wind}^{(1)}\#\,[\boldsymbol{P}(\boldsymbol{u}(\tilde{\gamma}))]=M$$

and

$$\mathrm{wind}^{(2)}\#\,[\boldsymbol{P}(\boldsymbol{u}(\tilde{\gamma}))]=N,\tag{5.5}$$

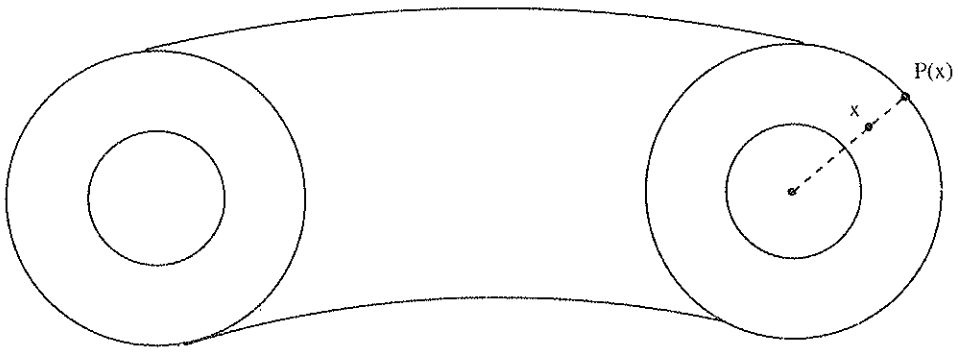

Figure 5.3. Vertical cross-section through Ω showing the action of $\boldsymbol{P}$.

for H^2 a.e. $\boldsymbol{x}=(x, y, z) \in T_b$ where $\tilde{\gamma}:[0,1] \to \Omega$ is defined by

$$\tilde{\gamma}(t)=c\left(\frac{x}{\tilde{r}}, \frac{y}{\tilde{r}}, 0\right)+(t(b-a)+a) \frac{1}{b}\left(x\left[1-\frac{c}{\tilde{r}}\right], y\left[1-\frac{c}{\tilde{r}}\right], z\right), \quad t \in[0,1]. \tag{5.6}$$

Remark 5.7. The projection is chosen so that $\boldsymbol{P}(\boldsymbol{u}(\tilde{\gamma}))$ is a closed curve on T_b. If $\boldsymbol{u} \in C(\bar{\Omega}) \cap \mathscr{A}$, then (5.5) holds for all $(x, y, z) \in T_b$ by Remark 5.5.

The proof of the next lemma is exactly analogous to that of Lemma 2.9, and is therefore omitted.

Lemma 5.8. *Let $(\boldsymbol{u}_n) \subset \mathscr{A}$ converge weakly to $\boldsymbol{u}$ in $W^{1,P}(\Omega)$, $P \geqq 1$. If $\boldsymbol{u}_n$ satisfies $(H_{M,N})$ for all $n \in \mathbb{N}$, then $\boldsymbol{u}$ satisfies $(H_{M,N})$.*

Existence of minimisers

Given $M, N \in \mathbb{Z}$, let

$$\mathscr{A}^{M,N}=\{\boldsymbol{u} \in \mathscr{A}: \boldsymbol{u} \text{ satisfies } (H_{M,N}), \det \nabla \boldsymbol{u}>0 \text{ almost everywhere}\},$$

where

$$E(\boldsymbol{u})=\int_\Omega W(\nabla \boldsymbol{u}(\boldsymbol{x}))\, dx,$$

and $(H_{M,N})$ is given by Definition 5.6. We assume throughout this subsection that the stored energy function $W: M^{3\times 3} \to \overline{\mathbb{R}}$ satisfies the following hypotheses:

(H1) W is continuous and $W(F)=\infty$ if and only if $\det F \leqq 0$;

(H2) W is polyconvex, i.e. $W(F)=g(F, \operatorname{adj} F, \det F)$ for all $F \in M_+^{3\times 3}$, where $g: M^{3\times 3} \times M^{3\times 3} \times (0, \infty) \to \mathbb{R}$ is convex;

(H3) $W(F) \geqq C_1(|F|^P+|\operatorname{adj} F|^Q)+C_2$ for all $F \in M^{3\times 3}$, $d \in (0, \infty)$, for some $P \geqq 2$ and $Q \geqq P/(P-1)$, where $C_1>0$ and C_2 are constants.

Theorem 5.9. *Let W satisfy* (H1), (H2), (H3); *then E attains a minimum on $\mathscr{A}^{M,N}$ for each $M, N \in \mathbb{Z}$.*

Proof. The proof of this theorem follows from Lemma 5.8 and [**4**, Theorem 6.1] and is analogous to the two-dimensional case (Theorem 3.1). □

REMARK 5.10. By exactly analogous arguments to those used in Remark 3.2, it follows that if $P > 3$ then, for each $M, N \in \mathbb{Z}$, the minimiser whose existence is given by Theorem 5.9 is a strong local minimiser of E in $W^{1,P}(\Omega) \cap L^\infty(\Omega)$.

6. Concluding remarks

In generalising the two-dimensional example of Section 3 to three dimensions, it is natural to try and replace the annulus A by a shell. However, there are serious difficulties in defining a homotopy condition corresponding to (H_N) in this three-dimensional setting. The fundamental topological problem that one encounters is that there is no smooth unit vector field defined on S^2 (the 'hairy ball theorem'), though there clearly is on S^1, and hence a notion of winding number on S^2 is not available. (Of course, there do exist examples of smooth unit vector fields on the torus and these give rise to the winding numbers $\text{wind}^{(i)}\# \gamma$ defined in the first part of Section 5.)

There are natural approaches to generalising the results in this paper to the case of mixed displacement traction problems. In this section, we outline one such approach and point out some difficulties that arise.

For simplicity, we focus attention on the following specific three-dimensional problem.

Let

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1, \ -L \leqq z \leqq L\} \tag{6.1}$$

be a cylinder of length $2L$. Given a deformation $\boldsymbol{u}:\Omega \to \mathbb{R}^3$, we specify the boundary displacement at the ends of the cylinder

$$\boldsymbol{u}|_{\partial\Omega_{\text{End}}} = \text{identity}, \tag{6.2}$$

where

$$\partial\Omega_{\text{End}} = \{(x, y, z) \in \bar{\Omega} : z = \pm L, \ x^2 + y^2 \leqq 1\}. \tag{6.3}$$

The displacement on the remainder of the boundary of the cylinder is left unspecified (so that in the subsequent variational problem this will give rise to a natural, zero traction, boundary condition).

Again, heuristically, one expects many equilibria to this mixed problem, corresponding, for example, to twisting one end of the cylinder through a multiple of 2π about the z-coordinate axis. (In Figure 6.1 we show two such deformations, together with the image of an axial line inscribed on the outer surface of Ω.)

The linking number

To analytically isolate the different classes of deformations sketched above, we recall the classical notion of the linking number of two non-coplanar, non-intersecting, closed C^1 curves $\gamma, \boldsymbol{\mu}:[0, 1] \to \mathbb{R}^3$, which is given by

$$\text{link}\#\,(\gamma, \boldsymbol{\mu}) = \frac{-1}{4\pi} \int_0^1 \int_0^1 \frac{\Lambda(s, t)}{|\gamma(s) - \boldsymbol{\mu}(t)|^3}\, ds\, dt,$$

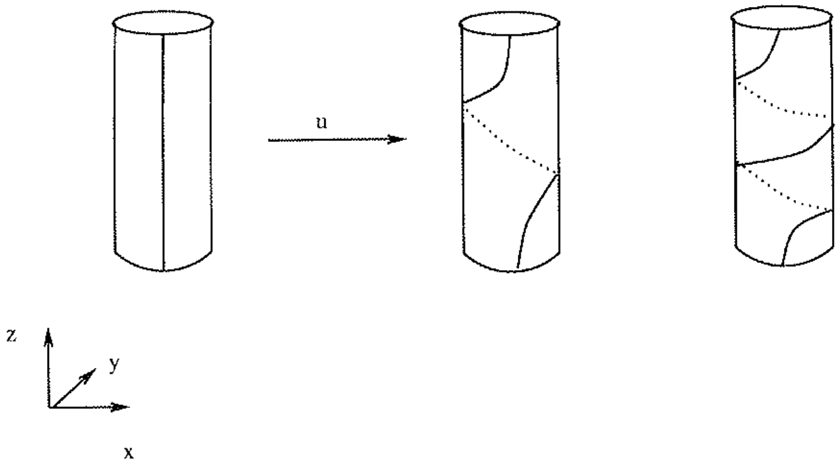

Figure 6.1

where Λ is the triple vector product

$$\Lambda(s, t) = [\dot{\gamma}(s) \wedge \dot{\mu}(t)] \cdot (\gamma(s) - \mu(t))$$

(see [**17**, p. 403]).

The notion of linking number is due originally to Gauss and is integer-valued. It counts the number of intersections of one of the curves with any surface spanning the other curve (taking into account multiplicity and direction of crossing, see [**17**, p. 403]); see Figure 6.2.

By the approximation arguments of Section 2, we can extend this definition of linking number to non-intersecting continuous curves γ and μ and show that it is well defined and locally continuous in the uniform topology on such curves. Hence if (γ_n), (μ_n) are closed continuous converging uniformly to γ and μ, respectively, then link# (γ_n, μ_n) is well defined for sufficiently large n and $\lim_{n\to\infty}$ link# $(\gamma_n, \mu_n) =$ link# (γ, μ).

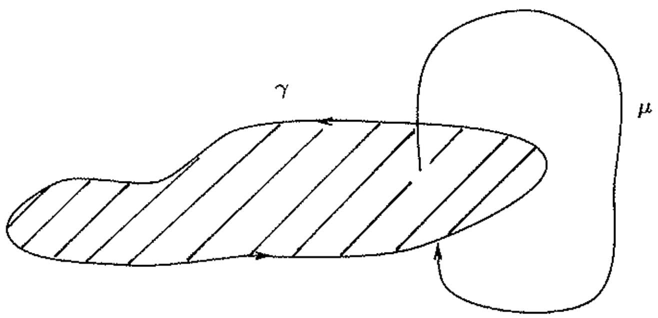

Figure 6.2. Two curves with link# $(\gamma, \mu) = 1$.

Classes of maps

Proceeding analogously to our approach earlier in this paper, we attempt to analytically identify these different classes of maps. We work in cylindrical polar coordinates (r, θ, z), where $r \in [0, 1]$, $\theta \in [0, 2\pi)$. For each $\bar{r} \in [0, 1)$, $\bar{\theta} \in [0, 2\pi)$, let $l_{\bar{r},\bar{\theta}}$ denote the straight line in Ω parallel to the z-axis given by $r = \bar{r}$, $\theta = \bar{\theta}$, $-L \leqq z \leqq L$.

We first identify the ends of the cylinder by joining them in the obvious way; i.e. we identify pairs of points (r, θ, L), $(r, \theta, -L)$ for each $(r, \theta) \in [0, 1] \times [0, 2\pi)$. Under this association, each axial line $l_{\bar{r},\bar{\theta}}$ gives rise to a corresponding closed curve denoted by $\gamma_{\bar{r},\bar{\theta}} : [-L, L] \to \mathbb{R}^3$ (parametrised by $z \in [-L, L]$). Let $\boldsymbol{\mu} : [-L, L] \to \mathbb{R}^3$ denote the closed curve corresponding to the choice $\bar{r} = 0$ (i.e. the axis of the cylinder).

Now given $N \in \mathbb{N}$ and a deformation $\boldsymbol{u} \in C(\bar{\Omega})$ satisfying (6.2), we say that $\boldsymbol{u}$ satisfies the condition $(\bar{H}_N)$ if link# $(\boldsymbol{u}(\gamma_{\bar{r},\bar{\theta}}), \boldsymbol{u}(\boldsymbol{\mu}))$ is defined and equal to N for all $(\bar{r}, \bar{\theta}) \in (0, 1] \times [0, 2\pi)$.

Existence of energy minimisers

Now, given $P > 3$, we consider the variational problem of minimising the stored energy functional given by (5) and (H1)–(H3) in the last part of Section 5 on the set of deformations

$$\tilde{\mathscr{A}} = \left\{ \boldsymbol{u} \in W^{1,p}(\Omega) : \det \nabla \boldsymbol{u} > 0 \text{ almost everywhere}, \right.$$

$$\left. \boldsymbol{u} = \text{identity on } \partial\Omega_{\text{End}}, \int_\Omega \det \nabla \boldsymbol{u}(x)\, dx \leqq \text{Vol } \boldsymbol{u}(\Omega) \right\}$$

The integral constraint in the definition of $\tilde{\mathscr{A}}$ is included to prevent interpenetration of matter by deformations in $\tilde{\mathscr{A}}$ (see [7]). Since $P > 3$, we may assume (by the Sobolev Embedding Theorem) without loss of generality that if $\boldsymbol{u} \in \tilde{\mathscr{A}}$ then $\boldsymbol{u} \in C(\bar{\Omega})$. By the results of [7], it follows that E attains a minimum on $\tilde{\mathscr{A}}$. We next consider minimising E on

$$\tilde{\mathscr{A}}^N = \{\boldsymbol{u} \in \tilde{\mathscr{A}} : \boldsymbol{u} \text{ satisfies } (\bar{H}_N)\},$$

where $N \in \mathbb{N}$ is given (and fixed).

At present, it is unclear to us under what (if any) hypotheses a minimising sequence $(\boldsymbol{u}_n) \subset \tilde{\mathscr{A}}^N$ necessarily has a subsequence converging weakly in $W^{1,p}(\Omega)$ to some $\boldsymbol{u} \in \tilde{\mathscr{A}}^N$. One problem is that there may exist (potential limit) deformations $\boldsymbol{u} \in W^{1,p}(\Omega)$, $\det \nabla \boldsymbol{u} > 0$ almost everywhere, $E(\boldsymbol{u}) < +\infty$ and such that link# $(\boldsymbol{u}(\gamma_{\bar{r},\bar{\theta}}), \boldsymbol{u}(\boldsymbol{\mu}))$ is undefined for any $(\bar{r}, \bar{\theta}) \in (0, 1] \times [0, 2\pi)$ as shown by the next example.

EXAMPLE 6.1. Let $W(F) = \alpha |F|^p + \beta (\det F)^r - \gamma \log |\det F|$ for $F \in M^{3\times 3}_+$ where $\alpha, \beta, \gamma > 0$ are constants, $p \geqq 1$, $r \geqq 1$. Now consider a deformation

$$\boldsymbol{u}(x) = (\varphi(z)x, \varphi(z)y, z) \quad \text{for } \boldsymbol{x} = (x, y, z) \in \Omega.$$

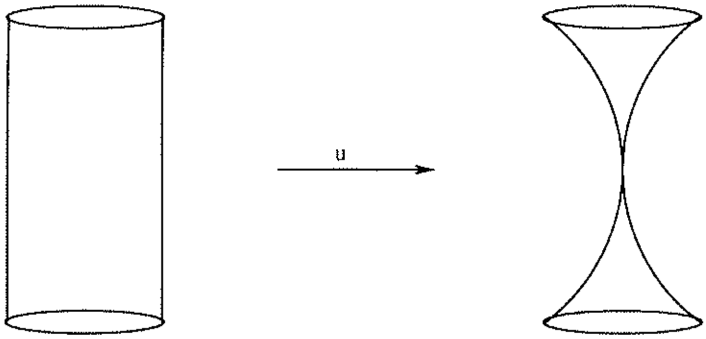

Figure 6.3

Then

$$\nabla \boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} \varphi & 0 & \varphi' x \\ 0 & \varphi & \varphi' y \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \det \nabla \boldsymbol{u} = \varphi^2(z).$$

Let $\varphi(z) = z^\alpha$ where $\alpha \geqq 1$.

Then an easy calculation yields $\boldsymbol{u} \in W^{1,p}(\Omega)$, $E(\boldsymbol{u}) < +\infty$, $\det \nabla \boldsymbol{u} > 0$ almost everywhere, but the linking number $\text{link}\#\,(\boldsymbol{u}(\gamma_{\bar{r},\bar{\theta}}), \boldsymbol{u}(\mu))$ is undefined for any $(\bar{r}, \bar{\theta}) \in (0, 1] \times [0, 2\pi)$. (See Fig. 6.3 for the case $\alpha = 2$.)

Indeed, deformations like $\boldsymbol{u}$ may be relevant in the modelling of fracture in certain materials (so that an energy minimising deformation $\boldsymbol{u}$ may develop such a singularity as the number $N \in \mathbb{N}$ is increased).

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References

1 J. M. Ball. Constitutive inequalities and existence theorems in nonlinear elastostatics. In *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium Vol. 1*, ed. R. J. Knops, 187–241 (London: Pitman, 1977).

2 J. M. Ball. Minimisers and the Euler–Lagrange equations. In *Proceedings of ISIMM Conference, Paris* (Berlin: Springer, 1983).

3 J. M. Ball, J. C. Currie and P. J. Olver. Null Lagrangians, weak continuity and variational problems of arbitrary order. *J. Funct. Anal.* **41** (1981), 135–74.

4 J. M. Ball and F. Murat. $W^{1,p}$-Quasiconvexity and variational problems for multiple integrals. *J. Funct. Anal.* **58** (1984), 225–53.

5 P. Baumann, N. C. Owen and D. Phillips. Maximum principles and *apriori* estimates for a class of problems from nonlinear elasticity. *Anal. Nonlineaire* **8** (1991), 119–57.

6 P. G. Ciarlet. *Mathematical Elasticity Vol. 1: Three-Dimensional Elasticity* (Amsterdam: North Holland, 1988).

7 P. G. Ciarlet and J. Necas. Injectivity and self-contact in nonlinear elasticity. *Arch. Rational Mech. Anal.* **97** (1987), 171–88.

8 L. C. Evans and R. F. Gariepy. *Measure Theory and the Fine Properties of Functions* (Boca Raton, Florida: CRC Press, 1992).
9 M. J. Greenberg. *Lectures on Algebraic Topology* (Reading, MA: W. A. Benjamin, 1967).
10 R. D. James and S. J. Spector. Remarks on $W^{1,p}$-quasiconvexity, interpenetration of matter, and function spaces for elasticity. *Ann. Inst. H. Poincaré, Anal. Non Linéaire* **9** (1992), 263–80.
11 F. John. Uniqueness of nonlinear equilibrium for prescribed boundary displacements and sufficiently small strains. *Comm. Pure Appl. Math.* **25** (1972), 617–34.
12 C. B. Morrey. *Multiple integrals in the calculus of variations* (Berlin: Springer, 1966).
13 S. Müller and S. J. Spector. An existence theory for nonlinear elasticity that allows for cavitation. *Arch. Rational Mech. Anal.* **131** (1995), 1–66.
14 R. W. Ogden. *Nonlinear Elastic Deformations* (Chichester: Ellis Horwood Ltd., Halstead Press, Wiley, 1984).
15 J. T. Schwartz. *Nonlinear Functional Analysis* (New York: Gordan and Breach, 1969).
16 J. Sivaloganathan. The generalised Hamilton–Jacobi inequality and the stability of equilibria in nonlinear elasticity. *Arch. Rational Mech. Anal.* **107** (1989), 347–69.
17 M. Spivak. *A Comprehensive Introduction to Differential Geometry, Vol. 1* (Berkeley, CA: Publish or Perish Inc, 1979).
18 M. Struwe. *Variational Methods* (Berlin: Springer, 1990).
19 V. Sverak. Regularity properties of deformations with finite energy. *Arch. Rational Mech. Anal.* **100** (1988), 105–27.
20 C. Truesdell and W. Noll. *The Non-Linear Field Theories of Mechanics*, Handbuch der Physik III/3 (Berlin: Springer, 1965).

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