The Representation Theorem for Linear, Isotropic Tensor Functions in Even Dimensions

Kathleen A. Pericak-Spector (kpericak@math.siu.edu)
Department of Mathematics, Southern Illinois University, Carbondale, IL 62901–4408, USA

Jeyabal Sivaloganathan (js@maths.bath.ac.uk)
Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK

Scott J. Spector (sspector@math.siu.edu)
Department of Mathematics, Southern Illinois University, Carbondale, IL 62901–4408, USA

Abstract. In this note we give a proof of the representation theorem for linear, isotropic, tensor functions, which only assumes invariance under *proper* orthogonal tensors.

Keywords: Linear Elasticity, Isotropic Tensors.

Mathematics Subject Classifications (2000): 74B05, 74A20.

1. Introduction

The well-known representation theorem for the elasticity tensor $\mathbb{C}$ of an isotropic body shows that there are real constants μ and λ such that for every $\mathbf{E} \in \mathrm{Sym}$, the space of symmetric tensors,

$$\mathbb{C}[\mathbf{E}] = 2\mu\mathbf{E} + \lambda(\mathrm{tr}\,\mathbf{E})\mathbf{Id}, \tag{1.1}$$

where $\mathrm{tr}\,\mathbf{E}$ denotes the trace of $\mathbf{E}$ and $\mathbf{Id}$ is the identity tensor. The standard proofs (see, e.g., [1–2, 5–6]) of this result start with the assumption that $\mathbb{C}$ is a symmetric, linear function (on Sym) that satisfies

$$\mathbb{C}[\mathbf{Q}\mathbf{E}\mathbf{Q}^T] = \mathbf{Q}\mathbb{C}[\mathbf{E}]\mathbf{Q}^T \tag{1.2}$$

for all $\mathbf{E} \in \mathrm{Sym}$ and all $\mathbf{Q} \in \mathrm{Orth}$, the set of orthogonal tensors (i.e., those tensors that satisfy $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{Id}$). However, the assumption that a body is composed of an *isotropic material*, as it is usually understood (see, e.g., [2], [7]), restricts the $\mathbf{Q}$ in (1.2) to be *proper*, viz., $\det\mathbf{Q} = +1$, where det denotes the determinant. (Such materials are called *hemitropic* in [1] and [8]). In 3-dimensions, or more generally in any odd-dimension, these assumptions are equivalent since $\mathbb{C}[(-\mathbf{Q})\mathbf{E}(-\mathbf{Q}^T)] = \mathbb{C}[\mathbf{Q}\mathbf{E}\mathbf{Q}^T]$ and $\det(-\mathbf{Q}) = -\det(\mathbf{Q})$. Thus, (1.2) will be satisfied for all orthogonal $\mathbf{Q}$ whenever it is satisfied by those $\mathbf{Q}$ that have positive determinant.

The only proof of the representation theorem, which we are aware of, that is valid in even dimensions and merely requires the invariance of the elasticity tensor under proper orthogonal mappings is in a recent paper by Jaric [3]. The purpose of this note is to provide an alternate proof of Jaric's result.

One way to interpret (1.1) (see, e.g., Knowles [4]) is that **Id** is an eigentensor of the linear symmetric mapping $\mathbb{C} : \mathrm{Sym} \to \mathrm{Sym}$ (with eigenvalue $2\mu + n\lambda$) and the orthogonal complement, Sym_0, the space of traceless symmetric tensors, is an eigenspace of $\mathbb{C}$ (with eigenvalue 2μ). The proof that **Id** is an eigentensor is well known and straightforward (see Proposition 2.2). The essential idea in many proofs of the representation theorem is to use the given invariance to show that Sym_0 is indeed an eigenspace of $\mathbb{C}$. A standard technique[1] to accomplish this is to use reflections in order to first show that $\mathbf{E}$ and $\mathbb{C}[\mathbf{E}]$ have the same eigenvectors, for each symmetric tensor $\mathbf{E}$ (see, e.g., the transfer theorem of Gurtin [2], pp. 231–232). However, reflections are not available in even dimensions if one only assumes invariance under the proper orthogonal group. Our approach uses rotations by 90° of two-dimensional subspaces instead of reflections. Such rotations allow us to show that $\mathbb{C}$ must map certain two-dimensional subspaces of Sym_0 into themselves. The spectral theorem for $\mathbb{C}$ restricted to such a subspace then yields a pair of (essentially) two-dimensional, traceless, symmetric eigentensors of $\mathbb{C}$. The existence of a single eigentensor of $\mathbb{C}$ of the form $\mathbf{e}\otimes\mathbf{e} - \mathbf{f}\otimes\mathbf{f}$, with $\mathbf{e}$ and $\mathbf{f}$ orthogonal unit vectors, then follows from the standard spectral theorem applied to either of the eigentensors of $\mathbb{C}$. Finally, we note (cf. Martins [5], Lemma 2) that the span of $\{\mathbf{Q}(\mathbf{e}\otimes\mathbf{e} - \mathbf{f}\otimes\mathbf{f})\mathbf{Q}^T : \mathbf{Q} \in \mathrm{Orth}^+\}$ is Sym_0, which together with the assumed invariance establishes Sym_0 is an eigenspace of $\mathbb{C}$.

2. Preliminaries

We let $\mathcal{V}$ denote a real n-dimensional vector space with inner product $\mathbf{a}\cdot\mathbf{b}$, $\mathbf{a},\mathbf{b} \in \mathcal{V}$, and write Lin for the space of all linear transformations (tensors) from $\mathcal{V}$ into $\mathcal{V}$. An inner product on Lin is given by

$$\mathbf{L} : \mathbf{M} := \mathrm{tr}(\mathbf{L}\mathbf{M}^T).$$

A subspace $\Upsilon \subset \mathrm{Lin}$ is said to be *invariant* (under Orth^+) provided that $\mathbf{Q}\mathbf{U}\mathbf{Q}^T \in \Upsilon$ whenever $\mathbf{U} \in \Upsilon$ and $\mathbf{Q} \in \mathrm{Orth}^+$. Examples of invariant subspaces of Lin are Sym and Sym_0. Let $\mathbb{A} : \Upsilon \to \Upsilon$ be a mapping of an invariant subspace $\Upsilon \subset \mathrm{Lin}$ into itself. Then we say that $\mathbb{A}$ is *isotropic* provided that for any $\mathbf{U} \in \Upsilon$

$$\mathbb{A}[\mathbf{Q}\mathbf{U}\mathbf{Q}^T] = \mathbf{Q}\mathbb{A}[\mathbf{U}]\mathbf{Q}^T \quad \text{for all} \quad \mathbf{Q} \in \mathrm{Orth}^+.$$

We denote by $\mathbf{a}\otimes\mathbf{b} \in \mathrm{Lin}$ the tensor product of $\mathbf{a},\mathbf{b} \in \mathcal{V}$:

$$(\mathbf{a}\otimes\mathbf{b})\mathbf{u} := (\mathbf{b}\cdot\mathbf{u})\mathbf{a} \ \text{ for } \ \mathbf{u} \in \mathcal{V};$$

We write $\mathbf{A}\boxtimes\mathbf{B} : \mathrm{Lin} \to \mathrm{Lin}$ for the tensor product of $\mathbf{A},\mathbf{B} \in \mathrm{Lin}$:

$$(\mathbf{A}\boxtimes\mathbf{B})\mathbf{U} := (\mathbf{B}:\mathbf{U})\mathbf{A} \ \text{ for } \ \mathbf{U} \in \mathrm{Lin}.$$

[1] An alternative idea, which is only valid in three-dimensions but does not use reflections, is to make use of the fact that nontrivial rotations in three-dimensions have a single (real) eigenvector, see, e.g., Ogden [7], p. 193.

Crucial to our results will be the spectral theorem.

PROPOSITION 2.1. *(Spectral Theorem.)*

(i) *Let* $\mathbf{S} \in \mathrm{Sym}$. *Then there exists scalars* $\lambda_i \in \mathbb{R}$ *and an orthonormal basis* $\{\mathbf{e}_i\}$ *of* $\mathcal{V}$, $i = 1, 2, 3, \ldots, n$, *such that*

$$\mathbf{S} = \sum_{i=1}^{n} \lambda_i \mathbf{e}_i \otimes \mathbf{e}_i.$$

(ii) *Let* $\Upsilon \subset \mathrm{Lin}$ *be an m-dimensional subspace. Suppose that* $\mathbb{L} : \Upsilon \to \Upsilon$ *is linear and symmetric, i.e.,* $\mathbf{A} : \mathbb{L}[\mathbf{B}] = \mathbf{B} : \mathbb{L}[\mathbf{A}]$ *for all* $\mathbf{A}, \mathbf{B} \in \Upsilon$. *Then there exists scalars* $\Lambda_i \in \mathbb{R}$ *and an orthonormal basis* $\{\mathbf{F}_i\}$ *of* Υ, $i = 1, 2, 3, \ldots, m$, *such that*

$$\mathbb{L} = \sum_{i=1}^{m} \Lambda_i \mathbf{F}_i \boxtimes \mathbf{F}_i.$$

We will require the following result which is well known, although we include a proof for the convenience of the reader.

PROPOSITION 2.2. *Let* $\mathbb{C} : \mathrm{Sym} \to \mathrm{Sym}$ *be linear, isotropic, and symmetric. Then*

(i) *there exists* $\omega \in \mathbb{R}$ *such that* $\mathbb{C}[\mathbf{Id}] = \omega \mathbf{Id}$;

(ii) $\mathbb{C}_0 := \mathbb{C}|_{\mathrm{Sym}_0}$ *is linear, isotropic, and symmetric and satisfies* $\mathrm{Range}(\mathbb{C}_0) \subset \mathrm{Sym}_0$.

Proof. (i). We first note that $\mathbf{S} := \mathbb{C}[\mathbf{Id}] \in \mathrm{Sym}$ and hence the spectral theorem yields an orthonormal basis of eigenvectors, $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, for $\mathbb{C}[\mathbf{Id}]$ and corresponding eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$. Next, let $\mathbf{Q} \in \mathrm{Orth}^+$. Then in view of the isotropy of $\mathbb{C}$,

$$\mathbb{C}[\mathbf{Id}] = \mathbb{C}[\mathbf{Q}\mathbf{Q}^T] = \mathbf{Q}\mathbb{C}[\mathbf{Id}]\mathbf{Q}^T.$$

Now, let j be any integer between 1 and n and define

$$\mathbf{Q}_j^{\pm} := \pm(\mathbf{e}_j \otimes \mathbf{e}_1) + (\mathbf{e}_1 \otimes \mathbf{e}_j) + \mathbf{P}_j,$$

where $\mathbf{P}_j = \mathbf{Id} - (\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_j \otimes \mathbf{e}_j)$ is the orthogonal projection onto $(\mathrm{span}\{\mathbf{e}_1, \mathbf{e}_j\})^{\perp}$. Then the argument used in the proof of Lemma 4.1 shows that $\det(\mathbf{Q}_j^- \mathbf{Q}_j^+) = -1$ for each j and hence exactly one of $\mathbf{Q}_j^+$ and $\mathbf{Q}_j^-$, which we denote by $\mathbf{Q}_j$, is contained in Orth^+.

We note that $\mathbf{Q}_j\mathbf{e}_j = \mathbf{e}_1$ (no summation), $\mathbf{Q}_j^T\mathbf{e}_1 = \mathbf{e}_j$, and consequently

$$\lambda_1\mathbf{e}_1 = \mathbb{C}[\mathbf{Id}]\mathbf{e}_1 = \mathbf{Q}_j\mathbb{C}[\mathbf{Id}]\mathbf{Q}_j^T\mathbf{e}_1 = \mathbf{Q}_j\mathbb{C}[\mathbf{Id}]\mathbf{e}_j = \lambda_j\mathbf{Q}_j\mathbf{e}_j = \lambda_j\mathbf{e}_1.$$

Thus all of the eigenvalues of the symmetric tensor $\mathbb{C}[\mathbf{Id}]$ are equal, which proves (i).

(ii). If $\mathbf{E} \in \mathrm{Sym}_0$ then, by (i) and the symmetry of $\mathbb{C}$, $\mathrm{tr}\,\mathbb{C}[\mathbf{E}] = \mathbf{Id} : \mathbb{C}[\mathbf{E}] = \mathbf{E} : \mathbb{C}[\mathbf{Id}] = \omega\mathbf{E} : \mathbf{Id} = \omega\,\mathrm{tr}\,\mathbf{E} = 0$. Thus $\mathrm{Range}(\mathbb{C}_0) \subset \mathrm{Sym}_0$. The linearity, isotropy, and symmetry of the restriction follow from the corresponding properties of $\mathbb{C}$.

□

3. Rank-two Eigentensors

PROPOSITION 3.1. *Let* $\mathbb{C}_0 : \mathrm{Sym}_0 \to \mathrm{Sym}_0$ *be linear, isotropic, and symmetric. Suppose that* $\mathbf{f}_1, \mathbf{f}_2 \in \mathcal{V}$ *are a pair of orthogonal unit vectors. Define the two-dimensional subspace*

$$\Upsilon_{12} := \mathrm{span}\{\mathbf{f}_1 \otimes \mathbf{f}_1 - \mathbf{f}_2 \otimes \mathbf{f}_2,\ \mathbf{f}_1 \otimes \mathbf{f}_2 + \mathbf{f}_2 \otimes \mathbf{f}_1\} \subset \mathrm{Sym}_0,$$

and the linear operator $\mathbb{L}_0 := \mathbb{C}_0|_{\Upsilon_{12}} : \Upsilon_{12} \to \mathrm{Sym}_0$. *Then* $\mathrm{Range}(\mathbb{L}_0) \subset \Upsilon_{12}$ *and hence there exist* $\Lambda_1, \Lambda_2 \in \mathbb{R}$ *and an orthonormal pair* $\mathbf{F}_1, \mathbf{F}_2 \in \Upsilon_{12}$ *such that*

$$\mathbb{L}_0 = \Lambda_1\mathbf{F}_1 \boxtimes \mathbf{F}_1 + \Lambda_2\mathbf{F}_2 \boxtimes \mathbf{F}_2.$$

Proof. We first note that once we establish that the range of $\mathbb{L}_0$ is Υ_{12} the desired result will follow from (ii) of the Spectral Theorem. In order to establish $\mathrm{Range}(\mathbb{L}_0) \subset \Upsilon_{12}$ define the plane

$$\mathcal{S} := \mathrm{span}\{\mathbf{f}_1, \mathbf{f}_2\} \subset \mathcal{V},$$

and the orthogonal projection $\mathbf{P} := \mathbf{Id} - (\mathbf{f}_1 \otimes \mathbf{f}_1 + \mathbf{f}_2 \otimes \mathbf{f}_2)$ onto $\mathcal{S}^\perp$.

We let $\mathbf{E} = \alpha(\mathbf{f}_1 \otimes \mathbf{f}_1 - \mathbf{f}_2 \otimes \mathbf{f}_2) + \beta(\mathbf{f}_1 \otimes \mathbf{f}_2 + \mathbf{f}_2 \otimes \mathbf{f}_1) \in \Upsilon_{12}$ and define $\mathbf{R} \in \mathrm{Orth}^+$ to be the rotation by angle $\frac{\pi}{2}$ in the $\mathcal{S}$-plane defined by $\mathbf{R} = \mathbf{f}_1 \otimes \mathbf{f}_2 - \mathbf{f}_2 \otimes \mathbf{f}_1 + \mathbf{P}$. Then we extend $\{\mathbf{f}_1, \mathbf{f}_2\}$ to an orthonormal basis relative to which we obtain the matrix representation

$$\mathbf{R} = \left[\begin{array}{cc|c} 0 & 1 & \mathbf{0} \\ -1 & 0 & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{I} \end{array}\right], \qquad \mathbf{E} = \left[\begin{array}{cc|c} \alpha & \beta & \mathbf{0} \\ \beta & -\alpha & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{0} \end{array}\right], \tag{3.1}$$

where $\mathbf{I}$ is the $(n-2) \times (n-2)$ identity matrix. For future reference we note that it is clear from $(3.1)_1$ that

$$\mathbf{R}\mathbf{a} = -\mathbf{a} \quad \Rightarrow \quad \mathbf{a} = \mathbf{0}. \tag{3.2}$$

Moreover, it follows from (3.1) that $\mathbf{RER}^T = -\mathbf{E}$ and hence, in view of the linearity and isotropy of $\mathbb{C}_0$,

$$\mathbf{R}\mathbb{C}_0[\mathbf{E}]\mathbf{R}^T = \mathbb{C}_0[\mathbf{RER}^T] = \mathbb{C}_0[-\mathbf{E}] = -\mathbb{C}_0[\mathbf{E}]. \tag{3.3}$$

Now, let $\mathbf{g} \in \mathcal{S}^\perp$. Then $\mathbf{R}^T\mathbf{g} = \mathbf{g}$ and thus by (3.3)

$$\mathbf{R}(\mathbb{C}_0[\mathbf{E}]\mathbf{g}) = -(\mathbb{C}_0[\mathbf{E}]\mathbf{g}).$$

However, in view of our previous observation (3.2)

$$\mathbb{C}_0[\mathbf{E}]\mathbf{g} = \mathbf{0} \quad \text{for every} \quad \mathbf{g} \in \mathcal{S}^\perp,$$

or, equivalently, since $\mathbb{C}_0[\mathbf{E}] \in \mathrm{Sym}$,

$$\mathbf{0} = \mathbb{C}_0[\mathbf{E}]\mathbf{g} = \left[\begin{array}{cc|c} \alpha' & \beta' & \mathbf{A} \\ \beta' & \gamma' & \\ \hline \multicolumn{2}{c|}{\mathbf{A}^T} & \mathbf{B} \end{array}\right] \left[\begin{array}{c} \mathbf{0} \\ \hline \mathbf{g}' \end{array}\right] = \left[\begin{array}{c} \mathbf{A}\mathbf{g}' \\ \hline \mathbf{B}\mathbf{g}' \end{array}\right],$$

for all $\mathbf{g}'$. Consequently, $\mathbf{Ag}' = \mathbf{0}$ and $\mathbf{Bg}' = \mathbf{0}$ for all $\mathbf{g}'$ and hence $\mathbf{A} = \mathbf{0}$ and $\mathbf{B} = \mathbf{0}$. Since $\mathbb{C}_0[\mathbf{E}]$ is also traceless $\gamma' = -\alpha'$ and thus $\mathbb{C}_0[\mathbf{E}] \in \Upsilon_{12}$. Since $\mathbf{E} \in \Upsilon_{12}$ is arbitrary this shows that $\mathrm{Range}(\mathbb{C}_0|_{\Upsilon_{12}}) \subset \Upsilon_{12}$. □

COROLLARY 3.2. *Let $\mathbb{C}_0 : \mathrm{Sym}_0 \to \mathrm{Sym}_0$ be linear, isotropic, and symmetric. Then there exists a pair of orthogonal unit vectors $\mathbf{g}_1, \mathbf{g}_2 \in \mathcal{V}$ and a scalar $\mu \in \mathbb{R}$ that satisfies*

$$\mathbb{C}_0[\mathbf{g}_1 \otimes \mathbf{g}_1 - \mathbf{g}_2 \otimes \mathbf{g}_2] = 2\mu[\mathbf{g}_1 \otimes \mathbf{g}_1 - \mathbf{g}_2 \otimes \mathbf{g}_2].$$

Proof. Let $\mathbf{f}_1, \mathbf{f}_2 \in \mathcal{V}$ be a pair of orthogonal unit vectors. Then, by Proposition 3.1, there exists a tensor $\mathbf{F}_1 \in \Upsilon_{12}$ and a scalar $\Lambda_1 \in \mathbb{R}$ that satisfies

$$\mathbb{C}_0[\mathbf{F}_1] = \Lambda_1\mathbf{F}_1. \tag{3.4}$$

Now, in view of (i) of the Spectral Theorem and the fact that $\mathbf{F}_1 \in \Upsilon_{12} \subset \mathrm{Sym}_0$ has norm one, there is an orthonormal pair $\mathbf{g}_1, \mathbf{g}_2 \in \mathcal{V}$ such that

$$\mathbf{F}_1 = \frac{1}{\sqrt{2}}(\mathbf{g}_1 \otimes \mathbf{g}_1 - \mathbf{g}_2 \otimes \mathbf{g}_2). \tag{3.5}$$

The desired result then follows from (3.4) and (3.5) with $\mu := \Lambda_1/2$. □

4. The Representation Theorem

LEMMA 4.1. *Let $\mathbf{g}_1, \mathbf{g}_2 \in \mathcal{V}$ and $\mathbf{h}_1, \mathbf{h}_2 \in \mathcal{V}$ each be a pair of orthogonal unit vectors. Then there exists $\mathbf{Q} \in \mathrm{Orth}^+$ that satisfies*

$$\mathbf{Qg}_1 = \pm\mathbf{h}_1, \quad \mathbf{Qg}_2 = \mathbf{h}_2.$$

Proof. Given orthonormal pairs $(\mathbf{g}_1, \mathbf{g}_2)$ and $(\mathbf{h}_1, \mathbf{h}_2)$, let $\{\mathbf{g}_i\}$ and $\{\mathbf{h}_i\}$, $i = 1, 2, 3, \ldots, n$, each be an orthonormal basis for $\mathcal{V}$. Then each of the tensors

$$\mathbf{Q}_\pm = \pm(\mathbf{h}_1 \otimes \mathbf{g}_1) + \sum_{i=2}^{n} \mathbf{h}_i \otimes \mathbf{g}_i$$

is orthogonal and, consequently, $\det \mathbf{Q}_+ = \pm 1$ and $\det \mathbf{Q}_- = \pm 1$. Moreover,

$$(\mathbf{Q}_-)^T \mathbf{Q}_+ = -\mathbf{g}_1 \otimes \mathbf{g}_1 + \sum_{i=2}^{n} \mathbf{g}_i \otimes \mathbf{g}_i = \left[\begin{array}{c|c} -1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right],$$

and therefore $(\det \mathbf{Q}_-)(\det \mathbf{Q}_+) = -1$. Thus, exactly one of $\mathbf{Q}_+$ and $\mathbf{Q}_-$ has positive determinant and is therefore contained in Orth^+. This is the desired $\mathbf{Q}$. □

THEOREM 4.2. *Let* $\mathbb{C}_0 : \mathrm{Sym}_0 \to \mathrm{Sym}_0$ *be linear, isotropic, and symmetric. Then there exists a scalar* $\mu \in \mathbb{R}$ *such that*

$$\mathbb{C}_0[\mathbf{E}] = 2\mu\mathbf{E} \quad \textit{for every} \quad \mathbf{E} \in \mathrm{Sym}_0.$$

Proof. We first note that by Corollary 3.2 there exists a pair of orthogonal unit vectors $\mathbf{g}_1, \mathbf{g}_2 \in \mathcal{V}$ and a scalar $\mu \in \mathbb{R}$ that satisfies

$$\mathbb{C}_0[\mathbf{g}_1 \otimes \mathbf{g}_1 - \mathbf{g}_2 \otimes \mathbf{g}_2] = 2\mu[\mathbf{g}_1 \otimes \mathbf{g}_1 - \mathbf{g}_2 \otimes \mathbf{g}_2]. \tag{4.1}$$

Now, let $\mathbf{E} \in \mathrm{Sym}_0$. Then by (i) of the Spectral Theorem there are scalars $\lambda_i \in \mathbb{R}$ and an orthonormal basis $\{\mathbf{e}_i\} \in \mathcal{V}$, $i = 1, 2, 3, \ldots, n$, such that

$$\mathbf{E} = \sum_{i=1}^{n} \lambda_i \mathbf{e}_i \otimes \mathbf{e}_i. \tag{4.2}$$

Moreover, since $\mathbf{E}$ is traceless

$$\lambda_n = -\sum_{i=1}^{n-1} \lambda_i. \tag{4.3}$$

Next, by Lemma 4.1, there exists $\mathbf{Q}_i \in \mathrm{Orth}^+$, $i = 1, 2, 3, \ldots, n-1$, such that $\mathbf{Q}_i \mathbf{g}_1 = \pm \mathbf{e}_i$, $\mathbf{Q}_i \mathbf{g}_2 = \mathbf{e}_{i+1}$, and hence

$$[\mathbf{e}_i \otimes \mathbf{e}_i - \mathbf{e}_{i+1} \otimes \mathbf{e}_{i+1}] = \mathbf{Q}_i [\mathbf{g}_1 \otimes \mathbf{g}_1 - \mathbf{g}_2 \otimes \mathbf{g}_2] \mathbf{Q}_i^T. \tag{4.4}$$

Since $\mathbb{C}_0$ is isotropic we find, using (4.1) and (4.4), that

$$\begin{aligned} \mathbb{C}_0[\mathbf{e}_i \otimes \mathbf{e}_i - \mathbf{e}_{i+1} \otimes \mathbf{e}_{i+1}] &= \mathbb{C}_0\left[\mathbf{Q}_i[\mathbf{g}_1 \otimes \mathbf{g}_1 - \mathbf{g}_2 \otimes \mathbf{g}_2]\mathbf{Q}_i^T\right] \\ &= \mathbf{Q}_i \mathbb{C}_0[\mathbf{g}_1 \otimes \mathbf{g}_1 - \mathbf{g}_2 \otimes \mathbf{g}_2]\mathbf{Q}_i^T \\ &= \mathbf{Q}_i 2\mu[\mathbf{g}_1 \otimes \mathbf{g}_1 - \mathbf{g}_2 \otimes \mathbf{g}_2]\mathbf{Q}_i^T \\ &= 2\mu[\mathbf{e}_i \otimes \mathbf{e}_i - \mathbf{e}_{i+1} \otimes \mathbf{e}_{i+1}]. \end{aligned} \tag{4.5}$$

Next, by (4.3) we find that

$$\mathbf{E} = \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i} \lambda_j \right) [\mathbf{e}_i \otimes \mathbf{e}_i - \mathbf{e}_{i+1} \otimes \mathbf{e}_{i+1}]$$

and hence, using (4.2), (4.5), and the linearity of $\mathbb{C}_0$, we obtain

$$\begin{aligned} \mathbb{C}_0[\mathbf{E}] &= \mathbb{C}_0 \left[\sum_{i=1}^{n-1} \left(\sum_{j=1}^{i} \lambda_j \right) [\mathbf{e}_i \otimes \mathbf{e}_i - \mathbf{e}_{i+1} \otimes \mathbf{e}_{i+1}] \right] \\ &= \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i} \lambda_j \right) \mathbb{C}_0[\mathbf{e}_i \otimes \mathbf{e}_i - \mathbf{e}_{i+1} \otimes \mathbf{e}_{i+1}] \\ &= \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i} \lambda_j \right) 2\mu[\mathbf{e}_i \otimes \mathbf{e}_i - \mathbf{e}_{i+1} \otimes \mathbf{e}_{i+1}] \\ &= 2\mu \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i} \lambda_j \right) [\mathbf{e}_i \otimes \mathbf{e}_i - \mathbf{e}_{i+1} \otimes \mathbf{e}_{i+1}] \;=\; 2\mu\mathbf{E}. \end{aligned}$$

□

COROLLARY 4.3. *Let* $\mathbb{C} : \mathrm{Sym} \to \mathrm{Sym}$ *be linear, isotropic, and symmetric. Then there exist scalars* $\lambda, \mu \in \mathbb{R}$ *such that*

$$\mathbb{C}[\mathbf{E}] = 2\mu\mathbf{E} + \lambda(\mathrm{tr}\,\mathbf{E})\mathbf{Id} \quad \textit{for every} \quad \mathbf{E} \in \mathrm{Sym}.$$

Proof. We first note that, by Proposition 2.2, there is a $\omega \in \mathbb{R}$ such that

$$\mathbb{C}[\mathbf{Id}] = \omega\,\mathbf{Id} \tag{4.6}$$

and, moreover, the restriction $\mathbb{C}_0 := \mathbb{C}|_{\mathrm{Sym}_0} : \mathrm{Sym}_0 \to \mathrm{Sym}_0$ is linear, isotropic, and symmetric. Consequently, by Theorem 4.2, there is a $\mu \in \mathbb{R}$

$$\mathbb{C}_0[\mathbf{E}_0] = 2\mu\mathbf{E}_0 \quad \text{for every} \quad \mathbf{E}_0 \in \mathrm{Sym}_0. \tag{4.7}$$

Now let $\mathbf{E} \in \mathrm{Sym}$ and define $\mathbf{E}_0 := \mathbf{E} - \frac{1}{n}(\mathrm{tr}\,\mathbf{E})\mathbf{Id}$. Then $\mathbf{E}_0 \in \mathrm{Sym}_0$ and hence, by (4.6), (4.7), and the linearity of $\mathbb{C}$

$$\begin{aligned} \mathbb{C}[\mathbf{E}] &= \mathbb{C}[\mathbf{E}_0 + \tfrac{1}{n}(\mathrm{tr}\,\mathbf{E})\mathbf{Id}] \\ &= \mathbb{C}[\mathbf{E}_0] + \tfrac{1}{n}(\mathrm{tr}\,\mathbf{E})\mathbb{C}[\mathbf{Id}] \\ &= 2\mu\mathbf{E}_0 + \tfrac{1}{n}(\mathrm{tr}\,\mathbf{E})\omega\mathbf{Id} \\ &= 2\mu(\mathbf{E} - \tfrac{1}{n}(\mathrm{tr}\,\mathbf{E})\mathbf{Id}) + \omega\tfrac{1}{n}(\mathrm{tr}\,\mathbf{E})\mathbf{Id} \\ &= 2\mu\mathbf{E} + \lambda(\mathrm{tr}\,\mathbf{E})\mathbf{Id}, \end{aligned}$$

where $\lambda := \frac{1}{n}(\omega - 2\mu)$. □

Acknowledgements

The research of SJS was supported by the National Science Foundation.

References

1. S. S. Antman, *Nonlinear Problems of Elasticity*, Springer, 1995.
2. M. E. Gurtin, *An Introduction to Continuum Mechanics*. Academic Press, 1981.
3. J. P. Jaric, On the representation of symmetric isotropic 4-tensors. *J. Elasticity* **51** (1998) 73-79.
4. J. K. Knowles, On the representation of the elasticity tensor for isotropic materials. *J. Elasticity* **39** (1995), 175–180.
5. L. C. Martins, The representation theorem for linear, isotropic tensor functions revisited. *J. Elasticity* **54** (1999), 89–92.
6. L. C. Martins and P. Podio-Guidugli, A new proof of the representation theorem for isotropic, linear constitutive relations. *J. Elasticity* **8** (1978), 319–322.
7. R. W. Ogden, *Non-linear Elastic Deformations*. Wiley, 1984.
8. C. Truesdell and W. Noll, The non-linear field theories of mechanics. In: S. Flügge (ed.), *Handbuch der Physik*, vol. III, Springer, 1972.