On the Existence of Minimizers with Prescribed Singular Points in Nonlinear Elasticity

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For Roger Fosdick, colleague, friend, and mentor

ABSTRACT. Experiments on elastomers have shown that sufficiently-large triaxial tensions induce the material to exhibit holes that were not previously evident. In this paper conditions are presented that allow one to use the direct method of the calculus of variations to deduce the existence of hole creating deformations that are global minimizers of a nonlinear, purely-elastic energy. The crucial physical assumption used is that there are a finite (possibly large) number of material points in the undeformed body that constitute the only points at which cavities can form. Each such point can be viewed as a preexisting flaw or an infinitesimal microvoid in the material.

1. Introduction

Experiments on elastomers have shown that holes, which were not previously evident to optical observation, appear in portions of the elastomer that experience a sufficiently-large triaxial tension. A reasonable expectation, especially in view of the analysis of Gent and Lindley [13] (see [18] for a brief description), is that such a cavitating deformation, i.e., an injective mapping that exhibits holes, should be a global minimizer of the elastic energy

$$E(\mathbf{u}) := \int_\Omega W(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x}))\, d\mathbf{x},$$

although it is possible that such a deformation is only a relative minimizer or that nonelastic effects, such as surface energy or plasticity, are needed to explain cavitation.

In this paper we present conditions on the stored energy density W that allow us to deduce the existence of a cavitating deformation that is a global minimizer of E among those deformations that are injective (almost everywhere) and satisfy sufficiently-large displacement boundary conditions on the boundary of a bounded region $\Omega \subset \mathbb{R}^3$.

The main physical assumption that we use to obtain our existence result is that there are a finite (possibly large) number of material points in the undeformed body that constitute the only points at which cavities can form. We view each such point as a preexisting flaw or weakness in the material (see [12]), however, one could just as well view it as an infinitesimal precursor microvoid (see [25] and Horgan and Abeyaratne [16]). Our theory assumes that such points are not detectable when the material is subjected to small (finite) deformations or purely compressive stresses. However, when any such point is subjected to a sufficiently-large, triaxial tension, a hole, which was not previously in the material, will be created at that point.

Our purely-elastic model may not explain the refined experiments of Cho and Gent [7]. These experiments indicate that a triaxial tension that induces a hole to appear in a large sample may not be sufficient to open a cavity when the same triaxial tension is only experienced in a small portion of the sample. In Section 4 we therefore also briefly consider a model that might better explain such results. In addition to the elastic energy E this model includes the energy of creation of each new hole. This creation energy may be different at each cavitation point and therefore model differing flaw strengths or preexisting microvoid sizes.

The main mathematical tools we use to prove our results are contained in [23]. These tools are based, in large part, on results on fine properties of deformations previously obtained by Šverák [27] and results on the distributional Jacobian previously obtained by Müller [22]. Our proof makes use of the notion of polyconvexity and the direct method of the calculus of variations, as first presented by John Ball [2] in the context of nonlinear elasticity, to deduce the existence of minimizers. Our approach to cavitation follows that of Ball [3] (see also [17] and the reference therein) who views cavitation as the creation of a new hole in the body rather than the growth of a preexisting finite hole, as had previously been analyzed in [13].

In addition to the above mentioned existence results we show, in Section 5, that energy minimizers satisfy various weak forms of the equilibrium equations. Therefore, since the cavitation points can be prescribed arbitrarily, our results imply that, for the stored-energy functions considered in this paper, there exist infinitely many singular weak solutions to the equations of nonlinear elastostatics. The proof of the results in this section follows the suggestion of Ball [4] that has been carried out in detail by Bauman, Owen, and Phillips [6] (see also Giaquinta, Modica, and Souček [14]). We also show that if the minimizer is smooth away from the finite number of prescribed cavitation points then, in the absence of body forces, the divergence of the Cauchy stress is zero in the deformed configuration and the divergence of the both the Piola-Kirchhoff stress and the the energy-momentum tensor are zero in the reference configuration.

An existence theorem that allows for cavitation has previously been obtained by Müller and Spector [23]. However, their model is not a purely-elastic one since it includes an additional energy that is proportional to the area of newly created cavity surface. Our model needs no such additional energy to obtain existence, but must instead predetermine the finite set of points at which cavitation may occur. The model in [23] makes no such prespecification. It also allows for a countable number of new holes to be created.

Finally, we mention that our paper has a second purpose. Suppose that the body, in its reference configuration, is a ball centered at the origin that is composed of a homogeneous, isotropic, nonlinearly elastic material. Then our intuition leads us to believe that, for many stored energy densities W, a global minimizer of the energy should be a radial deformation of the form

$$\mathbf{u}(\mathbf{x}) = \frac{r(|\mathbf{x}|)}{|\mathbf{x}|}\,\mathbf{x},$$

that minimizes the energy among radial, injective mappings and creates a single hole at the center of the ball. Unfortunately, except for an elastic fluid it has not been determined whether this conjecture is correct.[1] Thus an interesting, possibly simpler, open problem is to determine whether the radial minimizer is a global minimizer among injective deformations that open a single hole at the center. This paper establishes that the latter class contains a function that minimizes the energy within that class.

2. Preliminaries

In the following, Ω will denote a nonempty, bounded, open subset of $\mathbb{R}^n$, $n \geq 2$, whose boundary, $\partial\Omega$, is (strongly) Lipschitz (see Evans and Gariepy [10] or Morrey [21]). By $L^p(\Omega)$ and $W^{1,p}(\Omega)$ we denote the usual spaces of p-summable and Sobolev functions, respectively. We use the notation $L^p(\Omega;\mathbb{R}^m)$, etc., for vector-valued maps. A function is in $L^p_{\mathrm{loc}}(\Omega)$ if $\varphi \in L^p(U)$ for all open sets $U \subset\subset \Omega$, i.e., $U \subset K_U \subset \Omega$ for some compact set K_U. We point out that we *do not identify functions that agree almost everywhere*. We use the short-hand notation $\varphi \in W^{1,1}(\Omega)$, etc., to indicate that φ is a representative of an equivalence class that is contained in $W^{1,1}(\Omega)$. Weak convergence in these spaces will be indicated by the half arrow $\rightharpoonup$. Weak* (weak-star) convergence in $L^\infty(\Omega)$ will be denoted by $\overset{*}{\rightharpoonup}$ and the norm on this space will be denoted by $||\cdot||_\infty$. Composition of functions will be denoted by $\circ$.

The n-dimensional Lebesgue measure will be denoted by $\mathcal{L}^n$ and the k-dimensional Hausdorff measure by $\mathcal{H}^k$. We write

$$B(\mathbf{b}, r) := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{b}| < r\}$$

[1] Counterexamples have been obtained in [18]. However, these examples require W to grow so slowly at infinity that they are incompatible with the theory in this paper. See also [26] for a model problem in which the conjecture is true.

for the ball of radius r centered at $\mathbf{b} \in \mathbb{R}^n$ with volume $\omega_n r^n = \mathcal{L}^n(B(\mathbf{b}, r))$. For $\mathbf{b} \in \Omega$ we let

$$r_{\mathbf{b}} := \mathrm{dist}(\mathbf{b}, \partial\Omega),$$

i.e., the distance from $\mathbf{b}$ to the boundary of Ω.

We write Lin for the set of all linear maps from $\mathbb{R}^n$ into $\mathbb{R}^n$ with inner product and norm

$$\mathbf{K} \cdot \mathbf{L} = \mathrm{tr}(\mathbf{K}^T \mathbf{L}), \quad |\mathbf{L}| = (\mathbf{L} \cdot \mathbf{L})^{1/2}$$

respectively, where $\mathrm{tr}(\mathbf{M})$ denotes the trace of $\mathbf{M} \in \mathrm{Lin}$. We denote by $\mathrm{Lin}^>$ those $\mathbf{L} \in \mathrm{Lin}$ with positive determinant. The mapping adj : Lin $\to$ Lin will be the unique continuous function that satisfies

$$\mathbf{L}(\mathrm{adj}\,\mathbf{L}) = (\det \mathbf{L})\mathbf{Id}$$

for all $\mathbf{L} \in \mathrm{Lin}$, where $\det \mathbf{L}$ is the determinant of $\mathbf{L}$ and $\mathbf{Id} \in \mathrm{Lin}$ is the identity mapping. Thus, with respect to any orthonormal basis, the matrix corresponding to $\mathrm{adj}\,\mathbf{L}$ is the transpose of the cofactor matrix corresponding to $\mathbf{L}$. We denote by $\mathbf{id} : \mathbb{R}^n \to \mathbb{R}^n$ the identity mapping on $\mathbb{R}^n$, i.e., $\mathbf{id}(\mathbf{x}) \equiv \mathbf{x}$. We write div for the divergence operator in $\mathbb{R}^n$: for a vector field $\mathbf{u}$, $\mathrm{div}\,\mathbf{u} = \mathrm{tr}\nabla\mathbf{u}$; for a tensor field $\mathbf{S}$, $\mathrm{div}\,\mathbf{S}$ is the vector field with components $\Sigma_j \partial \mathrm{S}_{ij}/\partial x_j$.

We briefly recall some facts about the Brouwer degree (see, e.g., Fonseca and Gangbo [11] or Schwartz [24] for more details). Let $\mathbf{u} : \overline{\Omega} \to \mathbb{R}^n$ be a C^1 map. If $\mathbf{y}_0 \in \mathbb{R}^n \backslash \mathbf{u}(\partial\Omega)$ is such that $\det \nabla \mathbf{u}(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in \mathbf{u}^{-1}(\mathbf{y}_0)$, one defines

$$\deg(\mathbf{u}, \Omega, \mathbf{y}_0) := \sum_{\mathbf{x} \in \mathbf{u}^{-1}(\mathbf{y}_0)} \mathrm{sgn}\left[\det \nabla \mathbf{u}(\mathbf{x})\right]. \tag{2.1}$$

Thus, in particular, if $\mathbf{g} : \overline{\Omega} \to \mathbb{R}^n$ is a diffeomorphism with $\det \nabla \mathbf{g} > 0$ on Ω, then from (2.1) we conclude that

$$\deg(\mathbf{g}, \Omega, \mathbf{y}_0) = \begin{cases} 1, & \text{if } \mathbf{y}_0 \in \Omega \\ 0, & \text{if } \mathbf{y}_0 \in \mathbb{R}^n \backslash \overline{\Omega}. \end{cases} \tag{2.2}$$

If φ is a C^∞ function supported in the connected component of $\mathbb{R}^n \backslash \mathbf{u}(\partial\Omega)$ that contains $\mathbf{y}_0$, one can show that

$$\int_\Omega \varphi(\mathbf{u}(\mathbf{x})) \det \nabla \mathbf{u}(\mathbf{x})\, d\mathbf{x} = \deg(\mathbf{u}, \Omega, \mathbf{y}_0) \int_{\mathbb{R}^n} \varphi(\mathbf{y})\, d\mathbf{y}.$$

Using this formula and approximating by C^∞ functions, one can define $\deg(\mathbf{u}, \Omega, \mathbf{y})$ for any continuous function $\mathbf{u} : \Omega \to \mathbb{R}^n$ and any $\mathbf{y} \in \mathbb{R}^n \backslash \mathbf{u}(\partial\Omega)$. Moreover, the degree only depends on $\mathbf{u}|_{\partial\Omega}$. Accordingly we will henceforth write $\deg(\mathbf{u}, \partial\Omega, \mathbf{y})$ instead of $\deg(\mathbf{u}, \Omega, \mathbf{y})$. More generally, since a continuous function on a compact set can be extended to a continuous function on all of $\mathbb{R}^n$ one has the following result.

Proposition 2.1 *Let* $\overline{\mathbf{u}} \in C(\partial B(\mathbf{b}, r); \mathbb{R}^n)$ *for some* $\mathbf{b} \in \mathbb{R}^n$ *and* $r > 0$. *Then* $\deg(\overline{\mathbf{u}}, \partial B(\mathbf{b}, r), \mathbf{y})$ *is well defined for every* $\mathbf{y} \in \mathbb{R}^n \backslash \overline{\mathbf{u}}(\partial B(\mathbf{b}, r))$. *Moreover,*

(i) if $\overline{\mathbf{u}}_j \in C(\partial B(\mathbf{b}, r); \mathbb{R}^n)$ *is such that* $\overline{\mathbf{u}}_j \to \overline{\mathbf{u}}$ *uniformly then*

$$\lim_{j\to\infty} \deg(\overline{\mathbf{u}}_j, \partial B(\mathbf{b}, r), \mathbf{y}) = \deg(\overline{\mathbf{u}}, \partial B(\mathbf{b}, r), \mathbf{y})$$

for every $\mathbf{y} \in \mathbb{R}^n \backslash \overline{\mathbf{u}}(\partial B(\mathbf{b}, r))$;

(ii) if $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ *is a diffeomorphism that satisfies* $\det \nabla \mathbf{g} > 0$ *on* $\mathbb{R}^n$ *then*

$$\deg(\mathbf{g} \circ \overline{\mathbf{u}}, \partial B(\mathbf{b}, r), \mathbf{g}(\mathbf{y})) = \deg(\overline{\mathbf{u}}, \partial B(\mathbf{b}, r), \mathbf{y})$$

for every $\mathbf{y} \in \mathbb{R}^n \backslash \overline{\mathbf{u}}(\partial B(\mathbf{b}, r))$;

(iii) if $\mathbf{g} : \overline{\Omega} \to \overline{\Omega}$ *is a diffeomorphism that satisfies* $\det \nabla \mathbf{g} > 0$ *on* Ω *then*

$$\deg(\overline{\mathbf{u}} \circ \mathbf{g}, \partial B(\mathbf{b}, r), \mathbf{y}) = \deg(\overline{\mathbf{u}}, \partial B(\mathbf{b}, r), \mathbf{y})$$

for every $\mathbf{y} \in \mathbb{R}^n \backslash \overline{\mathbf{u}}(\partial B(\mathbf{b}, r))$.

Remarks. 1. In (i) one might have $\mathbf{y} \in \overline{\mathbf{u}}_j(\partial B(\mathbf{b}, r))$ and therefore $\deg(\overline{\mathbf{u}}_j, \partial B(\mathbf{b}, r), \mathbf{y})$ would not be defined. However, the uniform convergence $\overline{\mathbf{u}}_j \to \overline{\mathbf{u}}$ implies that the degree is well defined for all sufficiently large j.

2. For a ball one can find an explicit formula for the degree by using the extension $\mathbf{u} \in C^2(B(\mathbf{b}, r); \mathbb{R}^n) \cap C^0(\overline{B(\mathbf{b}, r)}; \mathbb{R}^n)$ that solves the vector Laplacian in $B(\mathbf{b}, r)$ with boundary values $\overline{\mathbf{u}}$. This function is defined by the Poisson integral formula (see, e.g., [15]).

3. Properties (ii) and (iii) are a consequence of (2.2) and the multiplicative property of the degree under compositions (see, e.g., Theorem 2.10 in [11], Theorem 3.20 in [24] or p. 578 in [29]).

Let $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^n)$, with $1 \leq p < n$. We will be interested in pointwise properties of $\mathbf{u}$ as well as restrictions of $\mathbf{u}$ to lower dimensional sets. In these cases, it is useful to consider an alternative representative in the same equivalence class. We define the **precise representative** $\mathbf{u}^* : \Omega \to \mathbb{R}^n$ by

$$\mathbf{u}^*(\mathbf{x}) = \begin{cases} \displaystyle\lim_{\rho\to 0^+} \fint_{B(\mathbf{x},\rho)} \mathbf{u}(\mathbf{z})\, d\mathbf{z}, & \text{if the limit exists,} \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where $\fint_A$ denotes the integral average over A, i.e., the integral of the function over A divided by the n-dimensional Lebesgue measure of A.

We shall make use of the fact that, if $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^n)$ with $1 \le p < n$, the above limit exists for every $\mathbf{x} \in \Omega \backslash P$, where $\mathcal{H}^{n-1}(P) = 0$. Thus, in particular, one can use the precise representative as a representative of the trace on $(n-1)$-dimensional surfaces. Moreover, if $p > n-1$ then $\mathcal{H}^1(P) = 0$ and consequently for each $\mathbf{b} \in \Omega$ the above limit is defined at *every* point on $\partial B(\mathbf{b}, r)$ for almost every $r \in (0, r_{\mathbf{b}})$. For a thorough discussion of precise representatives we refer to [10].

The following observation will be useful in our later development in this paper.

Proposition 2.2 (see, e.g., Lemma 2.9 in [23]). *Assume that*

$$\mathbf{u}_j \rightharpoonup \mathbf{u} \quad in \quad W^{1,p}(\Omega; \mathbb{R}^m).$$

Let $\mathbf{b} \in \Omega$ *and* $r_{\mathbf{b}} := \operatorname{dist}(\mathbf{b}, \partial\Omega)$. *Then there is an* $N_{\mathbf{b}} \subset \mathbb{R}$ *with* $\mathcal{L}^1(N_{\mathbf{b}}) = 0$ *such that for any* $r \in (0, r_{\mathbf{b}}) \backslash N_{\mathbf{b}}$ *there exists a subsequence* $\mathbf{u}_j$ *(not relabeled), which will in general depend on* r*, such that*

$$\mathbf{u}_j^* \rightharpoonup \mathbf{u}^* \quad in \quad W^{1,p}(\partial B(\mathbf{b}, r); \mathbb{R}^m).$$

Furthermore, if $p > n-1$*, then* $\mathbf{u}_j^*|_{\partial B(\mathbf{b},r)}$ *and* $\mathbf{u}^*|_{\partial B(\mathbf{b},r)}$ *are continuous and*

$$\mathbf{u}_j^* \to \mathbf{u}^* \quad uniformly\ on \quad \partial B(\mathbf{b}, r). \tag{2.3}$$

In nonlinear elasticity one is interested in globally invertible maps since, in general, matter cannot interpenetrate itself. We say that $\mathbf{u} \in W^{1,1}(\Omega; \mathbb{R}^n)$ is **invertible almost everywhere** (or equivalently, **one-to-one almost everywhere**) if there is a Lebesgue null set $N \subset \Omega$ such that $\mathbf{u}|_{\Omega \backslash N}$ is injective. We note that invertibility almost everywhere is a property of the equivalence class and not merely of the representative. However, the notion of invertibility almost everywhere is not sufficient for the analysis in function classes that allow for the formation of cavities. In fact the topological properties of such maps can differ drastically from everywhere invertible maps. The source of the difficulties is that a cavity formed at one point may be filled by material from elsewhere. In order to exclude such behavior the invertibility condition (INV) was introduced in [23].

Definition 2.3 *Let* $B(\mathbf{b}, r) \subset \Omega$ *and suppose that* $\overline{\mathbf{u}} : \partial B(\mathbf{b}, r) \to \mathbb{R}^n$ *is continuous. We define the* **topological image** *of* $B(\mathbf{b}, r)$ *under* $\overline{\mathbf{u}}$ *by*

$$\operatorname{im}_T(\overline{\mathbf{u}}, B(\mathbf{b}, r)) := \{\mathbf{y} \in \mathbb{R}^n \backslash \overline{\mathbf{u}}(\partial B(\mathbf{b}, r)) : \deg(\overline{\mathbf{u}}, \partial B(\mathbf{b}, r), \mathbf{y}) \neq 0\}.$$

Remark. If $\overline{\mathbf{u}} : \partial B(\mathbf{b}, r) \to \mathbb{R}^n$ happens to be the restriction of a homeomorphism $\mathbf{h} : \overline{B(\mathbf{b}, r)} \to \mathbb{R}^n$ to $\partial B(\mathbf{b}, r)$ then

$$\operatorname{im}_T(\overline{\mathbf{u}}, B(\mathbf{b}, r)) = \mathbf{h}(B(\mathbf{b}, r)).$$

Definition 2.4 *We say that* $\mathbf{u} : \Omega \to \mathbb{R}^n$ *satisfies condition (INV) provided that for every* $\mathbf{b} \in \Omega$ *there exists an* $\mathcal{L}^1$ *null set* $N_{\mathbf{b}}$ *such that, for all* $r \in (0, r_{\mathbf{b}})\backslash N_{\mathbf{b}}$, $\mathbf{u}|_{\partial B(\mathbf{b},r)}$ *is continuous,*

(i) $\mathbf{u}(\mathbf{x}) \in \mathrm{im}_T(\mathbf{u}, B(\mathbf{b}, r)) \cup \mathbf{u}(\partial B(\mathbf{b}, r))$ *for* $\mathcal{L}^n$ *a.e.* $\mathbf{x} \in \overline{B(\mathbf{b}, r)}$, *and*

(ii) $\mathbf{u}(\mathbf{x}) \in \mathbb{R}^n \backslash \mathrm{im}_T(\mathbf{u}, B(\mathbf{b}, r))$ *for* $\mathcal{L}^n$ *a.e.* $\mathbf{x} \in \Omega \backslash \overline{B(\mathbf{b}, r)}$.

Remarks. 1. Fix $\mathbf{b} \in \Omega$. Then one can think of (i) and (ii) as the requirement that (almost) every spherical shell centered at $\mathbf{b}$ is a *solid, impenetrable two-dimensional body* that is subjected to a continuous deformation. Thus, all matter that was originally inside such a shell must remain inside and all matter that was originally outside such a shell must remain outside.

2. Another way one might attempt to eliminate physically inappropriate maps is to require that each deformation $\mathbf{u}$ lie in the same homotopy class as the identity map. For example, one might require that there exists a mapping $\mathbf{g} \in C^0([0, 1]; W^{1,p}(\Omega; \mathbb{R}^n))$ with $\mathbf{g}(\cdot, t)$ one-to-one a.e. for every $t \in [0, 1]$, $\mathbf{g}(\mathbf{x}, 0) = \mathbf{x}$, and $\mathbf{g}(\mathbf{x}, 1) = \mathbf{u}(\mathbf{x})$ for a.e. $\mathbf{x} \in \Omega$. However, there are mathematical difficulties with such conditions. In particular, it is not clear that the weak limit of an energy minimizing sequence of deformations that lies in such a homotopy class also lies in the same homotopy class. In addition, one would need to show that the weak limit of a sequence of deformations that are one-to-one a.e. is also one-to-one a.e.

Deformations that satisfy condition (INV) and have nonzero Jacobian are more regular than other elements of the Sobolev spaces $W^{1,p}$, $n - 1 < p < n$. In particular, in [23] it is shown that such deformations have a representative that is continuous at every $\mathbf{x} \in \Omega\backslash\hat{P}$, where $\mathcal{H}^{n-p}(\hat{P}) = 0$ (and hence $\mathcal{L}^n(\hat{P}) = 0$) and, more significantly in the context of this work, are one-to-one a.e.

Proposition 2.5 (see Lemma 3.4 in [23]). *Let* $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^n)$ *with* $p > n - 1$. *Suppose that* $\det \nabla\mathbf{u} \neq 0$ *a.e. and that* $\mathbf{u}^*$ *satisfies condition (INV). Then* $\mathbf{u}$ *is one-to-one almost everywhere.*

For a diffeomorphism with nonnegative Jacobian the degree can only assume the values 1 and 0. The following is a counterpart of this result for the situation at hand.

Proposition 2.6 (see Lemma 3.5 in [23]). *Let* $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^n)$ *with* $p > n - 1$. *Suppose that* $\mathbf{u}^*$ *satisfies condition (INV) and that* $\det \nabla\mathbf{u} \neq 0$ *a.e. Fix* $\mathbf{b} \in \Omega$.

(i) Assume, in addition, that $\det \nabla \mathbf{u} > 0$ *a.e. Then there exists an* $\mathcal{L}^1$ *null set* $N_{\mathbf{b}}$ *such that for every* $r \in (0, r_{\mathbf{b}}) \backslash N_{\mathbf{b}}$

$$\deg(\mathbf{u}, \partial B(\mathbf{b}, r), \mathbf{y}) \in \{0, 1\}, \quad \textit{for all} \ \ \mathbf{y} \in \mathbb{R}^n \backslash \mathbf{u}(\partial B(\mathbf{b}, r)). \tag{2.4}$$

(ii) Conversely, if there is an $r_0 \in (0, r_{\mathbf{b}})$ *such that (2.4) is satisfied for* $\mathcal{L}^1$ *a.e.* $r \in (0, r_0)$ *then*

$$\det \nabla \mathbf{u} > 0 \quad \textit{a.e. in } B(\mathbf{b}, r_0).$$

Another result we will make use of is the change of variables formula for Sobolev mappings. (For a proof see Proposition 2.6 in [23] and the references therein).

Proposition 2.7 *Let* $\mathbf{u}$ *be (a representative of an equivalence class) in* $W^{1,1}(\Omega; \mathbb{R}^n)$*. Then there is a Lebesgue null set* $N_{\mathbf{u}} \subset \Omega$ *such that for any measurable function* $\varphi : \mathbb{R}^n \to \mathbb{R}$ *and any measurable set* $A \subset \Omega$

$$\int_A \varphi(\mathbf{u}(\mathbf{x})) |\det \nabla \mathbf{u}(\mathbf{x})| \, d\mathbf{x} = \int_{\mathbf{u}(A \backslash N_{\mathbf{u}})} \mathbb{N}(\mathbf{u}, A, \mathbf{y}) \varphi(\mathbf{y}) \, d\mathbf{y} \tag{2.5}$$

whenever either integral exists. Here

$$\mathbb{N}(\mathbf{u}, A, \mathbf{y}) := \text{cardinality } \{\mathbf{x} \in A : \mathbf{u}(\mathbf{x}) = \mathbf{y}\}.$$

Remark. Marcus and Mizel [19] showed that if $p > n$ and if $\mathbf{u}$ is (the continuous representative of an equivalence class) in $W^{1,p}(\Omega; \mathbb{R}^n)$ then $\mathbf{u}$ maps null sets onto null sets and one can replace $A \backslash N_{\mathbf{u}}$ by A in (2.5).

We call a (nonnegative outer) measure μ on Ω a *Radon measure* provided that every Borel subset of Ω is μ-measurable, every subset of Ω is contained in a Borel set of the same μ-measure, and every compact subset of Ω has finite μ-measure. Thus $\mathcal{L}^n$ is a Radon measure on Ω, but $\mathcal{H}^k$ is not when $k < n$. We say that a sequence of Radon measures μ_j on Ω converges *weak* in the sense of measures* to a Radon measure μ, denoted $\mu_j \overset{*}{\rightharpoonup} \mu$, provided

$$\int_\Omega \phi(\mathbf{x}) \, d\mu_j(\mathbf{x}) \to \int_\Omega \phi(\mathbf{x}) \, d\mu(\mathbf{x})$$

for every $\phi \in C_0(\Omega)$, i.e., every continuous function $\phi : \Omega \to \mathbb{R}$ that is supported in a compact subset of Ω. We will make use of the following result concerning Radon measures. For a proof, as well as a thorough discussion of Radon measures, see Evans and Gariepy [10].

Proposition 2.8 *(Weak* compactness of Radon measures). Let μ_j be a sequence of Radon measures on Ω that satisfy*

$$\sup_j \mu_j(K) < \infty \quad \textit{for each compact} \quad K \subset \Omega.$$

Then there exists a subsequence (not relabeled) and a Radon measure μ such that

$$\mu_j \overset{*}{\rightharpoonup} \mu \quad \textit{in the sense of measures.}$$

If $\mathbf{u} \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^n)$, with $p > n^2/(n+1)$, then the linear functional $(\text{Det}\,\nabla\mathbf{u}) : C^1_0(\Omega) \to \mathbb{R}$ given by

$$(\text{Det}\,\nabla\mathbf{u})(\phi) := -\frac{1}{n}\int_\Omega \nabla\phi \cdot (\text{adj}\,\nabla\mathbf{u})\mathbf{u}\,d\mathbf{x} \tag{2.6}$$

is a well-defined distribution, which is called the **distributional Jacobian**. If $\mathbf{u} \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^n)$, with $p \geq n$ then the identity $\text{div}(\text{adj}\,\nabla\mathbf{u})^T = \mathbf{0}$ can be used to show that $\text{Det}\,\nabla\mathbf{u}$ is the distribution induced by the function $\det\nabla\mathbf{u}$. Indeed, for simplicity let $\mathbf{u}$ be C^2 on $\overline{\Omega}$ then, by the product rule and the above identity,

$$\begin{aligned}\text{div}(\phi(\text{adj}\,\nabla\mathbf{u})\mathbf{u}) &= \nabla\phi \cdot (\text{adj}\,\nabla\mathbf{u})\mathbf{u} + \phi(\text{adj}\,\nabla\mathbf{u})^T \cdot \nabla\mathbf{u} + \phi\mathbf{u} \cdot \text{div}(\text{adj}\,\nabla\mathbf{u})^T \\ &= \nabla\phi \cdot (\text{adj}\,\nabla\mathbf{u})\mathbf{u} + n\phi(\det\nabla\mathbf{u})\end{aligned}$$

and hence, since ϕ has compact support, we conclude with the aid of the divergence theorem and (2.6) that

$$(\text{Det}\,\nabla\mathbf{u})(\phi) = \int_\Omega \phi(\mathbf{x})\det\nabla\mathbf{u}(\mathbf{x})\,d\mathbf{x}. \tag{2.7}$$

In general this need not be the case and in fact (2.7) will not be satisfied when cavitation occurs. For example, the deformation

$$\mathbf{w}(\mathbf{x}) = \frac{r(|\mathbf{x}|)}{|\mathbf{x}|}\mathbf{x}$$

with $r \in C([0,1]; \mathbb{R}) \cap C^1((0,1); \mathbb{R})$, $r(0) > 0$, and $r' > 0$, which creates a spherical hole at the center of the body $B(\mathbf{0}, 1)$, has

$$(\text{Det}\,\nabla\mathbf{w})(\phi) = \int_\Omega \phi(\mathbf{x})\det\nabla\mathbf{w}(\mathbf{x})\,d\mathbf{x} + \omega_n r(0)^n\phi(\mathbf{0}) \quad \text{for all} \quad \phi \in C^1_0(\Omega), \tag{2.8}$$

where ω_n is the volume of the n-dimensional unit ball $B(\mathbf{0}, 1)$.

The above paragraph shows that, for radial cavitation, *the distributional Jacobian is a Radon measure*, that is, there exists a (nonnegative) Radon (outer) measure $\mu_{_J}$ on Ω such that

$$(\text{Det}\,\nabla\mathbf{u})(\phi) = \int_\Omega \phi(\mathbf{x})\,d\mu_{_J}(\mathbf{x}) \quad \text{for all} \quad \phi \in C^1_0(\Omega). \tag{2.9}$$

In particular, if $\mathrm{Det}\,\nabla\mathbf{u} \geq 0$, i.e., $(\mathrm{Det}\,\nabla\mathbf{u})(\phi) \geq 0$ for all $\phi \in C_0^1(\Omega)$ that are nonnegative, then the Riesz representation theorem (see, e.g., pp. 49–54 in [10]) can be used to show that there exists a Radon measure μ_J that satisfies (2.9). To simplify notation, whenever such a measure exists we will denote it by $\mathrm{Det}\,\nabla\mathbf{u}$ and thus not distinguish between the distribution $\mathrm{Det}\,\nabla\mathbf{u}$ and the measure μ_J. Thus, for any open set $U \subset \Omega$,

$$(\mathrm{Det}\,\nabla\mathbf{u})(U) := \mu_J(U) := \sup\{(\mathrm{Det}\,\nabla\mathbf{u})(\phi) : \phi \in C_0^1(U), ||\phi||_\infty \leq 1\},$$

while, for an arbitrary set $A \subset \Omega$,

$$(\mathrm{Det}\,\nabla\mathbf{u})(A) := \mu_J(A) := \inf\{(\mathrm{Det}\,\nabla\mathbf{u})(U) : A \subset U \subset \Omega,\ U \text{ open}\}.$$

For example, we will write

$$\mathrm{Det}\,\nabla\mathbf{w} = (\det\nabla\mathbf{w})\mathcal{L}^n + \omega_n r(0)^n \delta_{\mathbf{0}}$$

instead of (2.8). Here, and in what follows, $\delta_{\mathbf{b}}$ denotes the Dirac measure centered at the point $\mathbf{b} \in \mathbb{R}^n$.

Now suppose that $\mathbf{u} \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$, with $p > n-1$ (rather than $p > n^2/(n+1)$). Then the precise representative $\mathbf{u}^*$ is continuous on the sphere $\partial B(\mathbf{b}, r)$ for almost every r and hence $\mathbf{u}^*(\partial B(\mathbf{b}, r))$ is compact for such r. If, in addition, $\mathbf{u}^*$ satisfies condition (INV) then it follows that $\mathbf{u}^* \in L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ and hence that (2.6) is once again a well-defined distribution on Ω. The next result shows that in fact this distribution is a Radon measure.

Proposition 2.9 (see Müller [22] and Lemma 8.1 in [23]). *Let* $\mathbf{u} \in W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ *with* $p > n-1$. *Suppose that* $\det\nabla\mathbf{u} > 0$ *a.e. and that* $\mathbf{u}^*$ *satisfies condition (INV). Then* $\mathrm{Det}\,\nabla\mathbf{u} \geq 0$ *and hence* $\mathrm{Det}\,\nabla\mathbf{u}$ *is a Radon measure. Furthermore,*

$$\mathrm{Det}\,\nabla\mathbf{u} = (\det\nabla\mathbf{u})\mathcal{L}^n + m,$$

where m is singular with respect to Lebesgue measure, and for $\mathcal{L}^1$ a.e. $r \in (0, r_{\mathbf{b}})$ one has

$$(\mathrm{Det}\,\nabla\mathbf{u})(\overline{B(\mathbf{b}, r)}) = \mathcal{L}^n(\mathrm{im}_T(\mathbf{u}, B(\mathbf{b}, r))). \tag{2.10}$$

3. Main Convergence Result

We now fix a finite set of points in the material and restrict our attention to deformations that may only open new holes in the interior of the body at these points.

Definition 3.1 *Let $p > n-1$ and suppose that $\mathbf{a}_i \in \Omega$, $i = 1, 2, 3, ..., M$ are given. Define* $\mathrm{Def}^p(\Omega, \mathbf{a}_1, ..., \mathbf{a}_M)$ *to be those maps* $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^n)$ *that satisfy:*

(i) $\mathbf{u}^*$ *satisfies condition (INV);*

(ii) $\det \nabla \mathbf{u} > 0$ *a.e.;*

(iii) $\mathrm{Det}\, \nabla \mathbf{u} = (\det \nabla \mathbf{u})\mathcal{L}^n + \sum_{i=1}^M \alpha^i \delta_{\mathbf{a}_i}$,

where α^i *are nonnegative real numbers that may depend on* $\mathbf{u}$.

Thus a deformation is a Sobolev mapping with positive Jacobian that can only open holes at a finite number of specified points in the body and which satisfies condition (INV) (and consequently is one-to-one a.e.).

Remarks. 1. Results in [23] imply that each deformation $\mathbf{u} \in \mathrm{Def}^{\,p}(\Omega, \mathbf{a}_1, ..., \mathbf{a}_M)$ has a representative that, in addition to being continuous $\mathcal{H}^{n-p}$ a.e., maps sets of measure zero onto such sets. This representative satisfies (i) and (ii) of condition (INV) for *every* $\mathbf{x} \in B(\mathbf{b}, r)$ and, when one uses this representative, $A \backslash N_{\mathbf{u}}$ can be replaced by A in equation (2.5) of the change of variables formula.

2. Note that deformations in $\mathrm{Def}^{\,p}(\Omega, \mathbf{a}_1, ..., \mathbf{a}_M)$ may not be smooth or even continuous on the entire complement of (potential) cavitation points $\mathbf{a}_i$. See Tang [28] for an example of a deformation that is not continuous yet opens no new holes.

3. Examples in [23] show that a deformation in $\mathrm{Def}^{\,p}(\Omega, \mathbf{a}_1, ..., \mathbf{a}_M)$ may open additional holes at the boundary of the region.

We now state the main result of this section.

Lemma 3.2 *Let* $p > n-1$ *and suppose that* $\mathbf{u}_j \in \mathrm{Def}^{\,p}(\Omega, \mathbf{a}_1, ..., \mathbf{a}_M)$ *satisfy*

$$\mathbf{u}_j \rightharpoonup \mathbf{u} \quad \text{in} \quad W^{1,p}(\Omega; \mathbb{R}^n) \tag{3.1}$$

and

$$\det \nabla \mathbf{u}_j \rightharpoonup \theta \quad \text{in} \quad L^1(\Omega) \tag{3.2}$$

for some $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^n)$ *and* $\theta \in L^1(\Omega)$*. Assume, further, that* $\theta > 0$ *a.e. in* Ω*. Then*

$$\mathbf{u} \in \mathrm{Def}^{\,p}(\Omega, \mathbf{a}_1, ..., \mathbf{a}_M) \quad \text{and} \quad \theta = \det \nabla \mathbf{u}. \tag{3.3}$$

In order to prove the above lemma we will need the following result. This result shows that the set of maps whose precise representatives satisfy (INV) is sequentially weakly closed in $W^{1,p}$. In addition, mappings that satisfy condition (INV) are in $L^\infty_{\mathrm{loc}}(\Omega)$ and consequently sequences of such deformations that converge weakly satisfy an additional convergence property.

Lemma 3.3 *Let $p > n-1$ and suppose that $\mathbf{u}_j^*$ is a sequence in $W^{1,p}(\Omega;\mathbb{R}^n)$ that satisfies condition (INV). Assume that*

$$\mathbf{u}_j \rightharpoonup \mathbf{u} \quad in \quad W^{1,p}(\Omega;\mathbb{R}^n).$$

Then $\mathbf{u}^*$ satisfies condition (INV). Moreover, there exists a subsequence (not relabeled) that satisfies

$$\mathbf{u}_j \to \mathbf{u} \quad in \quad L^q_{\text{loc}}(\Omega;\mathbb{R}^n) \tag{3.4}$$

for every $1 \le q < \infty$.

Proof of Lemma 3.3. Without loss of generality we take $\mathbf{u}_j = \mathbf{u}_j^*$ and $\mathbf{u} = \mathbf{u}^*$. For a proof that $\mathbf{u}$ satisfies (INV) see Lemma 3.3 in [23]. To prove (3.4) we first note that, by the Rellich compactness theorem, there is a subsequence that converges strongly to $\mathbf{u}$ in $L^p_{\text{loc}}(\Omega;\mathbb{R}^n)$ and hence a further subsequence (not relabeled) that satisfies $\mathbf{u}_j \to \mathbf{u}$ a.e. Let $1 \le q < \infty$, $\mathbf{b} \in \Omega$, and suppose that $r \in (0, r_\mathbf{b})$ is such that $\mathbf{u}$ satisfies (i) and (ii) of condition (INV) on $B(\mathbf{b}, r)$ and (2.3) of Proposition 2.2. We will show that

$$\mathbf{u}_j \to \mathbf{u} \quad \text{in} \quad L^q(B(\mathbf{b}, r);\mathbb{R}^n).$$

This will imply the desired result since any compact subset of Ω can be covered by a finite number of such balls.

Define

$$U := \{\mathbf{y} \in \mathbb{R}^n : \text{dist}\,(\mathbf{y}, \mathbf{u}\,(\partial B(\mathbf{b}, r))) \le 1\}\,.$$

Then by (2.3) of Proposition 2.2 one has that, for all $\mathbf{z} \in \partial B(\mathbf{b}, r)$ and for all j sufficiently large, $|\mathbf{u}_j(\mathbf{z}) - \mathbf{u}(\mathbf{z})| < 1$ and hence $\mathbf{u}_j(\mathbf{z}) \in U$. Thus for all such j the boundaries of each of the open sets $\text{im}_T(\mathbf{u}_j, B(\mathbf{b}, r))$ are contained in the bounded set U. Therefore, the sets $\text{im}_T(\mathbf{u}_j, B(\mathbf{b}, r))$ are bounded, uniformly in j, and hence, in view of (i) of (INV), there is a $K > 0$ such that

$$||\mathbf{u}_j||_{L^\infty(B(\mathbf{b},r))} \le K$$

for all j sufficiently large. Since $\mathbf{u}_j \to \mathbf{u}$ a.e., the desired result now follows from the Lebesgue dominated convergence theorem. □

Proof of Lemma 3.2. Without loss of generality we take $\mathbf{u}_j = \mathbf{u}_j^*$ and $\mathbf{u} = \mathbf{u}^*$. Let $\mathbf{u}_j \in \text{Def}^{\,p}(\Omega, \mathbf{a}_1, ..., \mathbf{a}_M)$ satisfy (3.1) and (3.2) where $\theta > 0$ a.e. Then, by (3.1) and Lemma 3.3, $\mathbf{u}^*$ satisfies condition (INV) and

$$\mathbf{u}_j \to \mathbf{u} \quad \text{in} \quad L^q_{\text{loc}}(\Omega;\mathbb{R}^n) \text{ for every } \; q \in (1, \infty). \tag{3.5}$$

Moreover, since $p > n-1$ (see, e.g., Theorem 3.4 in [1] or Theorem 7.5-1 in [8])

$$\text{adj}\,\nabla\mathbf{u}_j \rightharpoonup \text{adj}\,\nabla\mathbf{u} \quad \text{in} \quad L^{\frac{p}{n-1}}(\Omega;\mathbb{R}^n). \tag{3.6}$$

We now show that $\operatorname{Det}\nabla\mathbf{u}$ is a Radon measure. By (3.5) and (3.6)

$$(\operatorname{adj}\nabla\mathbf{u}_j)\mathbf{u}_j \rightharpoonup (\operatorname{adj}\nabla\mathbf{u})\mathbf{u} \quad \text{in} \quad L^1_{\text{loc}}(\Omega;\mathbb{R}^n) \tag{3.7}$$

and hence

$$(\operatorname{Det}\nabla\mathbf{u}_j)(\varphi) \to (\operatorname{Det}\nabla\mathbf{u})(\varphi) \quad \text{for every} \quad \varphi\in C_0^\infty(\Omega). \tag{3.8}$$

Next, let $K\subset\Omega$ be compact. Then there exists $\psi_K\in C_0^1(\Omega;[0,1])$ with $\psi_K=1$ on K. Let $K'\subset\Omega$ be the closure of the set where ψ_K is strictly positive. If we write μ_j for the measure $\operatorname{Det}\nabla\mathbf{u}_j$ we find, with the aid of (2.9), that

$$\begin{aligned}(\operatorname{Det}\nabla\mathbf{u}_j)(K) &= \int_K 1\,d\mu_j(\mathbf{x})\\ &\le \int_\Omega \psi_K(\mathbf{x})\,d\mu_j(\mathbf{x}) = (\operatorname{Det}\nabla\mathbf{u}_j)(\psi_K)\end{aligned}$$

and, by definition of the distributional Jacobian,

$$\begin{aligned}(\operatorname{Det}\nabla\mathbf{u}_j)(\psi_K) &:= -\frac{1}{n}\int_\Omega \nabla\psi\cdot(\operatorname{adj}\nabla\mathbf{u}_j)\mathbf{u}_j\,d\mathbf{x}\\ &\le n^{-1}\sup\{|\nabla\psi(\mathbf{x})| : \mathbf{x}\in K'\}||(\operatorname{adj}\nabla\mathbf{u}_j)\mathbf{u}_j||_{L^1(K')}.\end{aligned}$$

However, in view of (3.7), $||(\operatorname{adj}\nabla\mathbf{u}_j)\mathbf{u}_j||_{L^1(K')}$ is bounded, uniformly in j, and hence

$$\sup_j\,(\operatorname{Det}\nabla\mathbf{u}_j)(K) < +\infty \quad \text{for every compact} \quad K\subset\Omega. \tag{3.9}$$

Therefore, by the weak* compactness result for measures (Proposition 2.8), a subsequence of these measures converges to a Radon measure. This, together with (3.8), shows that $\operatorname{Det}\nabla\mathbf{u}$ is a Radon measure and

$$\operatorname{Det}\nabla\mathbf{u}_j \overset{*}{\rightharpoonup} \operatorname{Det}\nabla\mathbf{u} \quad \text{in the sense of measures.} \tag{3.10}$$

Next, since $\mathbf{u}_j\in \operatorname{Def}^p(\Omega,\mathbf{a}_1,...,\mathbf{a}_M)$

$$\operatorname{Det}\nabla\mathbf{u}_j = (\det\nabla\mathbf{u}_j)\mathcal{L}^n + \sum_{i=1}^M \alpha_j^i\delta_{\mathbf{a}_i}$$

with $\det\nabla\mathbf{u}_j>0$ a.e. and $\alpha_j^i\ge 0$. In particular

$$\alpha_j^i \le (\operatorname{Det}\nabla\mathbf{u}_j)\left(\overline{B(\mathbf{a}_i,r)}\right).$$

Thus, by (3.9) with $K:=\overline{B(\mathbf{a}_i,r)}$, for any k

$$0\le\alpha_k^i\le\sup_j\alpha_j^i\le\sup_j\,(\operatorname{Det}\nabla\mathbf{u}_j)(K)<+\infty.$$

Therefore there exist $\alpha^i \in [0,\infty)$ such that (for a subsequence) $\alpha^i_j \to \alpha^i$ and consequently, as $j \to \infty$,

$$\alpha^i_j \delta_{\mathbf{a}_i} \overset{*}{\rightharpoonup} \alpha^i \delta_{\mathbf{a}_i} \quad \text{in the sense of measures.}$$

In view of (3.2) we therefore conclude that

$$\operatorname{Det} \nabla \mathbf{u}_j \overset{*}{\rightharpoonup} \theta \mathcal{L}^n + \sum_{i=1}^{M} \alpha^i \delta_{\mathbf{a}_i} \quad \text{in the sense of measures,}$$

which together with (3.10) yields

$$\operatorname{Det} \nabla \mathbf{u} = \theta \mathcal{L}^n + \sum_{i=1}^{M} \alpha^i \delta_{\mathbf{a}_i}. \tag{3.11}$$

If $p > n^2/(n+1)$ we are done since a result of Müller [22] then yields $\theta = \det \nabla \mathbf{u}$, which together with (3.11) and the hypothesis $\theta > 0$ a.e. give (ii) and (iii) of Definition 3.1. In general, the desired result will follow if we show that $\det \nabla \mathbf{u} > 0$ a.e. since we can then apply Proposition 2.9 together with (3.11) to arrive at the same conclusion for all $p > n - 1$.

In order to get $\det \nabla \mathbf{u} > 0$ a.e. we first show that $\det \nabla \mathbf{u} \neq 0$ a.e. By the change of variables formula for Sobolev mappings (Proposition 2.7) and the fact that each $\mathbf{u}_j$ is one-to-one a.e. (Proposition 2.5) one has that for all $\varphi \in C_0(\mathbb{R}^n)$

$$\int_\Omega \varphi(\mathbf{u}_j(\mathbf{x})) \det \nabla \mathbf{u}_j(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{R}^n} \varphi(\mathbf{y}) \mathcal{X}_{\mathbf{u}_j(\Omega \setminus N_j)}(\mathbf{y})\, d\mathbf{y}, \tag{3.12}$$

for some Lebesgue null sets N_j.

In preparation to letting $j \to \infty$ in (3.12) we first note that, by the Rellich compactness theorem, we may assume that $\mathbf{u}_j \to \mathbf{u}$ a.e. by passing, if necessary, to a subsequence. Thus, since φ is continuous,

$$\varphi(\mathbf{u}_j(\mathbf{x})) \to \varphi(\mathbf{u}(\mathbf{x})) \ \text{ for a.e.} \quad \mathbf{x} \in \Omega. \tag{3.13}$$

Moreover, each of the compositions, $\varphi \circ \mathbf{u}_j$, is contained in $L^\infty(\Omega)$ since φ is bounded. Next, the sequence of characteristic functions satisfies $||\mathcal{X}_{\mathbf{u}_j(\Omega\setminus N_j)}||_\infty \leq 1$ and hence, since the unit ball is compact in the weak* topology, there is a $\zeta \in L^\infty(\mathbb{R}^n)$ with $||\zeta||_\infty \leq 1$ and a subsequence (not relabeled) such that

$$\mathcal{X}_{\mathbf{u}_j(\Omega\setminus N_j)} \overset{*}{\rightharpoonup} \zeta \quad \text{in} \quad L^\infty(\Omega). \tag{3.14}$$

We now take the limit of (3.12) as $j \to \infty$ to conclude, with the aid of (3.13), (3.14), and Lemma A.1, that

$$\int_\Omega \varphi(\mathbf{u}(\mathbf{x}))\theta(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{R}^n} \varphi(\mathbf{y})\zeta(\mathbf{y})\, d\mathbf{y}. \tag{3.15}$$

Equation (3.15) is satisfied for all continuous φ with compact support and hence, by approximation (using the monotone convergence theorem) for φ the characteristic function of any open set.

We now show that $\det \nabla\mathbf{u} \neq 0$ a.e. Let $N_{\mathbf{u}}$ be the Lebesgue null set from the change of variables formula (Proposition 2.7) and define

$$M := \{\mathbf{x} \in \Omega : \det \nabla\mathbf{u}(\mathbf{x}) = 0\}\backslash N_{\mathbf{u}}.$$

Then Proposition 2.7 (with $A = M$) yields

$$0 = \int_{\mathbf{u}(M)} \mathbb{N}(\mathbf{u}, M, \mathbf{y})\psi(\mathbf{y})\, d\mathbf{y}$$

for all measurable functions ψ and consequently $\mathbf{u}(M)$ is a null set. Let $\varepsilon > 0$ and suppose that $U \supset \mathbf{u}(M)$ is open with $\mathcal{L}^n(U) < \varepsilon$. Then $\mathcal{X}_{\mathbf{u}(M)} \leq \mathcal{X}_U$ and hence

$$\mathcal{X}_M(\mathbf{x}) \leq \mathcal{X}_{\mathbf{u}(M)}(\mathbf{u}(\mathbf{x})) \leq \mathcal{X}_U(\mathbf{u}(\mathbf{x})) \tag{3.16}$$

for a.e. $\mathbf{x} \in \Omega$. We note that $||\zeta||_\infty \leq 1$ and $\theta > 0$ a.e. and apply (3.15) with $\varphi = \mathcal{X}_U$ to obtain, with the aid of (3.16),

$$\int_M \theta(\mathbf{x})\, d\mathbf{x} \leq \int_\Omega \mathcal{X}_U(\mathbf{u}(\mathbf{x}))\theta(\mathbf{x})\, d\mathbf{x} = \int_U \zeta(\mathbf{y})\, d\mathbf{y} \leq \mathcal{L}^n(U) < \varepsilon.$$

Thus, since $\theta > 0$ a.e. and since ε was arbitrary, we deduce that $\mathcal{L}^n(M) = 0$. Therefore $\det \nabla\mathbf{u} \neq 0$ a.e. since $N_{\mathbf{u}}$ is itself a Lebesgue null set.

Finally, we show that $\det \nabla\mathbf{u} > 0$ a.e. Let $\mathbf{b} \in \Omega$ and let $N_{\mathbf{b}}$ be the $\mathcal{L}^1$ null set of Proposition 2.2. Then by Proposition 2.6(i) there exist $\mathcal{L}^1$ null sets N_j such that for every $r \in (0, r_{\mathbf{b}})\backslash N_j$

$$\deg(\mathbf{u}_j, \partial B(\mathbf{b}, r), \mathbf{y}) \in \{0, 1\}, \quad \text{for all } \ \mathbf{y} \in \mathbb{R}^n\backslash\mathbf{u}_j(\partial B(\mathbf{b}, r)). \tag{3.17}$$

Define $N := N_{\mathbf{b}} \cup (\cup_{j=1}^\infty N_j)$. Fix $r \in (0, r_{\mathbf{b}})\backslash N$. Then by Proposition 2.2 there is a subsequence (not relabeled) such that $\mathbf{u}_j \to \mathbf{u}$ uniformly on $\partial B(\mathbf{b}, r)$. Therefore, in view of (3.17) and Proposition 2.1, we conclude that

$$\deg(\mathbf{u}, \partial B(\mathbf{b}, r), \mathbf{y}) \in \{0, 1\}, \quad \text{for all } \ \mathbf{y} \in \mathbb{R}^n\backslash\mathbf{u}(\partial B(\mathbf{b}, r)). \tag{3.18}$$

Since (3.18) is satisfied for all $r \in (0, r_{\mathbf{b}})\backslash N$ and since $\det \nabla\mathbf{u} \neq 0$ a.e. we can apply Proposition 2.6(ii) to conclude that $\det \nabla\mathbf{u} > 0$ a.e. in $B(\mathbf{b}, r_{\mathbf{b}})$, for every $\mathbf{b} \in \Omega$. The desired result, (3.3), now follows from Proposition 2.9 and Equation (3.11). □

4. Existence of Minimizers

We consider an elastic body that, for convenience, we identify with the bounded, open, connected region $\mathcal{B} \subset \mathbb{R}^n$ with (strongly) Lipschitz boundary that it occupies in a fixed reference configuration.

4.1. The Displacement Problem

In order to address the displacement problem we let $\mathcal{B} \subset\subset \Omega \subset \mathbb{R}^n$, where Ω is bounded, open, and connected and has (strongly) Lipschitz boundary. We suppose that a diffeomorphism $\mathbf{d} : \overline{\Omega} \to \mathbb{R}^n$ with strictly positive Jacobian is given. If $\mathbf{u} \in W^{1,p}(\mathcal{B};\mathbb{R}^n)$ satisfies $\mathbf{u} - \mathbf{d} \in W_0^{1,p}(\mathcal{B};\mathbb{R}^n)$ then we define its *extension* $\mathbf{u}^e : \Omega \to \mathbb{R}^n$ by

$$\mathbf{u}^e(\mathbf{x}) := \begin{cases} \mathbf{u}(\mathbf{x}), & \mathbf{x} \in \mathcal{B}, \\ \mathbf{d}(\mathbf{x}), & \mathbf{x} \notin \mathcal{B}, \end{cases}$$

and note that $\mathbf{u}^e \in W^{1,p}(\Omega;\mathbb{R}^n)$.

We first ignore any possible energy due to hole formation (see the Introduction), fix $\mathbf{a}_k \in \mathcal{B}$, $k = 1, 2, 3, ..., M$, and seek a minimizer for the total elastic energy

$$E(\mathbf{u}) := \int_{\mathcal{B}} W(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x}))\, d\mathbf{x} \tag{4.1}$$

in the class of admissible functions

$$\mathcal{A}^p := \{\mathbf{u} \in W^{1,p}(\mathcal{B};\mathbb{R}^n) : \mathbf{u} - \mathbf{d} \in W_0^{1,p}(\mathcal{B};\mathbb{R}^n), \mathbf{u}^e \in \mathrm{Def}^{\,p}(\Omega, \mathbf{a}_1, ..., \mathbf{a}_M)\}.$$

Remarks. 1. The seemingly artificial requirement that admissible deformations possess local extensions that satisfy condition (INV) is necessitated by the problem of cavitation at the boundary (see [23]). In particular, given $\mathbf{a} \in \partial\mathcal{B}$ and (sufficiently small) $\alpha > 0$ one can construct deformations that satisfy

$$\mathrm{Det}\,\nabla\mathbf{u} = (\det\nabla\mathbf{u})\mathcal{L}^n, \quad \mathrm{Det}\,\nabla\mathbf{u}^e = (\det\nabla\mathbf{u}^e)\mathcal{L}^n + \alpha\delta_{\mathbf{a}}.$$

Thus the distributional Jacobian of a Sobolev mapping does not detect holes created at the boundary of the domain on which the mapping is defined. In the next subsection we will prove existence without such an extension requirement.

2. Note that an admissible mapping $\mathbf{u} \in \mathcal{A}^p$ must lie in $L^\infty(\Omega;\mathbb{R}^n)$ and satisfy $||\mathbf{u}||_\infty \leq ||\mathbf{d}||_\infty$.

Theorem 4.1 *Let $n = 3$, $p > 2$, $\mathcal{D} := \mathrm{Lin}^3 \times \mathrm{Lin}^3 \times (0,\infty)$, and suppose that the following conditions are satisfied:*

(i) (polyconvexity) *There is a function* $\Phi : \mathcal{B} \times \mathbb{R}^3 \times \mathcal{D} \to \mathbb{R}$ *such that, for every* $\mathbf{u} \in \mathbb{R}^3$ *and a.e.* $\mathbf{x} \in \mathcal{B}$,

$$W(\mathbf{x}, \mathbf{u}, \mathbf{F}) = \Phi(\mathbf{x}, \mathbf{u}, (\mathbf{F}, \operatorname{adj} \mathbf{F}, \det \mathbf{F})) \text{ whenever } \det \mathbf{F} > 0,$$

where $\Phi(\mathbf{x}, \mathbf{u}, \cdot) : \mathcal{D} \to \mathbb{R}$ *is convex for the same* $\mathbf{u}$ *and* $\mathbf{x}$.

(ii) (continuity) $\Phi(\mathbf{x}, \cdot, \cdot) : \mathbb{R}^3 \times \mathcal{D} \to \mathbb{R}$ *is continuous for a.e.* $\mathbf{x} \in \mathcal{B}$ *and* $\Phi(\cdot, \mathbf{u}, \mathcal{N}) : \mathcal{B} \to \mathbb{R}$ *is measurable for every* $(\mathbf{u}, \mathcal{N}) \in \mathbb{R}^3 \times \mathcal{D}$.

(iii) (coercivity) *For each compact set* $K \subset \mathbb{R}^3$

$$W(\mathbf{x}, \mathbf{u}, \mathbf{F}) \geq \phi_K(\mathbf{x}) + c|\mathbf{F}|^p + \Gamma(\det \mathbf{F})$$

for a.e. $\mathbf{x} \in \mathcal{B}$ *and every* $(\mathbf{u}, \mathbf{F}) \in K \times \mathrm{Lin}^3$ *with* $\det \mathbf{F} > 0$, *where* $c > 0$, $\phi_K \in L^1(\mathcal{B})$ *may depend on* K, *and* $\Gamma : (0, \infty) \to \mathbb{R}$ *is a convex function that satisfies* $\Gamma(t)/t \to +\infty$ *as* $t \to +\infty$.

(iv) $\Gamma(t) \to +\infty$ *as* $t \to 0^+$.

Then E *attains its infimum on* $\mathcal{A}^p$.

Remarks. 1. Body forces have been included in the function W. In particular, the dead-load body force $\mathbf{b}_0 \in L^1(\mathcal{B}; \mathbb{R}^3)$ would contribute a term of the form $\beta(\mathbf{x}, \mathbf{u}(\mathbf{x})) = \mathbf{b}_0(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x})$ to W. Such a term is usually not included in W due to difficulties in satisfying the coercivity hypothesis. However, in view of Remark 2 above, there is no such difficulty for the displacement problem since

$$|\beta(\mathbf{x}, \mathbf{u}(\mathbf{x}))| \leq |\mathbf{b}_0(\mathbf{x})| \, ||\mathbf{d}||_\infty.$$

2. Suppose that the boundary deformation is given by $\mathbf{d}(\mathbf{x}) = \lambda \mathbf{x}$. Then results of Ball [3] and many others (see [17]) imply that, for a large class of stored-energy functions of slow growth and for sufficiently large λ, any minimizer of E in the class $\mathcal{A}^p$ has strictly less energy than those deformations in $\mathcal{A}^p$ that satisfy $\mathrm{Det}\, \nabla \mathbf{u}^e = (\det \nabla \mathbf{u}^e)\mathcal{L}^3$. In other words, under these conditions the minimizer(s) of E will exhibit cavitation.

Proof. Since $\mathbf{d} \in \mathcal{A}^p$ the set $\mathcal{A}^p$ is nonempty. Further, by (ii), $E(\mathbf{d})$ is finite and hence so is the infimum. We note that $\mathbf{u} \in \mathcal{A}^p$ satisfies $||\mathbf{u}||_\infty \leq ||\mathbf{d}||_\infty$ and therefore choose $K := \overline{B(\mathbf{0}, ||\mathbf{d}||_\infty)}$. Thus, the coercivity of W implies that E is bounded below.

Let $\mathbf{u}_j \in \mathcal{A}^p$ be a minimizing sequence. Then by the coercivity of W and the Poincaré inequality (see, e.g., [21]) the sequence is bounded in $W^{1,p}(\mathcal{B}; \mathbb{R}^3)$ and hence we may assume (for a subsequence) that $\mathbf{u}_j \rightharpoonup \mathbf{u}$ in $W^{1,p}(\mathcal{B};\mathbb{R}^3)$, $\mathbf{u}_j \to \mathbf{u}$ in $L^p(\mathcal{B};\mathbb{R}^3)$, and $\mathbf{u}_j \to \mathbf{u}$ a.e. As usual we take $\mathbf{u}_j = \mathbf{u}_j^*$ and $\mathbf{u} = \mathbf{u}^*$.

In order to show that $\mathbf{u} \in \mathcal{A}^p$ we first note that that $(\mathbf{u}_j - \mathbf{d}) \in W_0^{1,p}(\mathcal{B};\mathbb{R}^3)$ and consequently $(\mathbf{u} - \mathbf{d}) \in W_0^{1,p}(\mathcal{B};\mathbb{R}^3)$. In particular, $\mathbf{u}^e \in W^{1,p}(\Omega;\mathbb{R}^3)$ is well-defined. Next, In view of the superlinear growth of Γ at $+\infty$ we find, by the de la Vallée Poussin and Dunford-Pettis criteria (see, e.g., [20]) that there is a $\theta \in L^1(\mathcal{B})$ such that (for a subsequence)

$$\det \nabla \mathbf{u}_j \rightharpoonup \theta \text{ in } L^1(\mathcal{B}).$$

Define $\theta^e \in L^1(\Omega)$ by

$$\theta^e(\mathbf{x}) := \begin{cases} \theta(\mathbf{x}), & \mathbf{x} \in \mathcal{B}, \\ \det \nabla \mathbf{d}(\mathbf{x}), & \mathbf{x} \notin \mathcal{B}, \end{cases}$$

and note that

$$\det \nabla \mathbf{u}_j^e \rightharpoonup \theta^e \text{ in } L^1(\Omega).$$

Clearly $\theta \geq 0$ a.e. If θ was equal to zero on a set $A \subset \mathcal{B}$ of positive ($\mathcal{L}^3$) measure then we would have $\det \nabla \mathbf{u}_j \to 0$ in $L^1(A)$ and hence, by hypothesis (iv), (for a subsequence) $\Gamma(\det \nabla \mathbf{u}_j(\mathbf{x})) \to +\infty$ for a.e. $\mathbf{x} \in A$. In this case we would have $E(\mathbf{u}_j) \to +\infty$, by the coercivity of W and Fatou's lemma. Therefore $\theta > 0$ a.e. and hence $\theta^e > 0$ a.e. Consequently, we can apply Lemma 3.2 to conclude that

$$\det \nabla \mathbf{u}_j^e \rightharpoonup \det \nabla \mathbf{u}^e \quad \text{in} \quad L^1(\Omega)$$

and $\mathbf{u}^e \in \mathrm{Def}^{\,p}(\Omega, \mathbf{a}_1, ..., \mathbf{a}_M)$. Therefore $\mathbf{u} \in \mathcal{A}^p$ and

$$\det \nabla \mathbf{u}_j \rightharpoonup \det \nabla \mathbf{u} \quad \text{in} \quad L^1(\mathcal{B}). \tag{4.2}$$

Finally, we show that $\mathbf{u}$ is a minimizer of E. We note that, since $p > 2$, $\operatorname{adj} \nabla \mathbf{u}_j \rightharpoonup \operatorname{adj} \nabla \mathbf{u}$ in $L^1(\mathcal{B}; \mathrm{Lin}^3)$ (see, e.g., Theorem 3.4 in [1] or Theorem 7.5-1 in [8]) and hence, in view of (4.2) and hypotheses (i) – (iii), we can apply the lower-semicontinuity theorem of Ball, Currie, and Olver (Theorem 5.4 in [5]) to conclude that

$$E(\mathbf{u}) \leq \liminf_{j\to\infty} E(\mathbf{u}_j) = \inf_{\mathcal{A}^p} E.$$

Therefore, $\mathbf{u} \in \mathcal{A}^p$ is the desired minimizer. □

As mentioned in the Introduction, our purely-elastic model may not explain the refined experiments of [7]. We now examine the effect of including an energy associated with the formation of a new hole in order to address this difficulty. Let $\eta_i \geq 0$ for $i = 1, ..., M$ and let $\upsilon : [0,\infty)^M \to [0,\infty)$ be lower semicontinuous. Define

$$I(\mathbf{u}) := E(\mathbf{u}) + \upsilon(\alpha^1, ..., \alpha^M) + \sum_{i=1}^{M} H(\alpha^i)\eta_i, \tag{4.3}$$

where

$$H(t) = \begin{cases} 1, & \text{if } \ t > 0 \\ 0, & \text{if } \ t \leq 0. \end{cases}$$

The term $H(\alpha^i)\eta_i$ represents the energy associated with initiating a cavity at the point $\mathbf{a}_i$. Its value may be different at each cavitation point due to differing strengths of the preexisting flaws (sizes of the preexisting microvoids) in the material. The term $\upsilon(\alpha^1, ..., \alpha^M)$ represents an energy associated with cavity growth. It might, for example, only be positive when the holes are small so as to measure the relatively-large surface energy needed to change a microvoid into a visible hole. Note that results on radial cavitation lead us to expect that the *cavitation load*, i.e., the minimal value of the principal stretches required to induce cavitation at a point, is an increasing function of the value of η at the point. A sufficiently-small portion of the material might only have precurers with large η, which could explain the results of [7].

We note that the proofs of Lemma 3.2 and Theorem 4.1 show that, for any ($\mathcal{L}^3$) measurable set $A \subset \mathcal{B}$, the mapping

$$\mathbf{u} \mapsto (\mathrm{Det}\, \nabla \mathbf{u})(A)$$

is sequentially weakly continuous on (an appropriately chosen subsequence of) each minimizing sequence in $\mathcal{A}^p$. In particular the choice $A = \{\mathbf{a}_i\}$ yields the sequential weak continuity of each of the the maps $\mathbf{u} \mapsto \alpha^i$. Since both υ and H are lower semicontinuous it follows that each of the terms that constitute I is sequentially weakly lower semicontinuous on minimizing sequences in $\mathcal{A}^p$. The proof of Theorem 4.1 therefore yields the following result.

Theorem 4.2 *Let $n = 3$, $p > 2$ and suppose that W satisfies the hypotheses of Theorem 4.1. Assume further that I is given by (4.3), where $\eta_i \geq 0$ for $i = 1, ..., M$ and $\upsilon : [0, \infty)^M \to [0, \infty)$ are lower semicontinuous. Then I attains its infimum on $\mathcal{A}^p$.*

Remark. The above result remains valid (see [23]) if one adds to I a surface energy that is proportional to the deformed surface area of the new cavities created in the material.

4.2. The Mixed Problem

In order to address the mixed problem let $\Omega = \mathcal{B}$ and $\partial\mathcal{B}_d \subset \partial\mathcal{B}$ with strictly positive (n-1)-dimensional measure ($\mathcal{H}^{n-1}(\partial\mathcal{B}_d) > 0$). We suppose that the deformation $\mathbf{d} : \partial\mathcal{B}_d \to \mathbb{R}^n$ is given on $\partial\mathcal{B}_d$, while dead-load tractions are prescribed on the remainder of the boundary, $\partial\mathcal{B}_t := \partial\mathcal{B}\backslash\partial\mathcal{B}_d$. We seek a minimizer for each of the energies

$$\begin{aligned} \hat{E}(\mathbf{u}) &:= E(\mathbf{u}) - L\mathbf{u}, \\ \hat{I}(\mathbf{u}) &:= I(\mathbf{u}) - L\mathbf{u}, \end{aligned}$$

where

$$L\mathbf{u} := \int_{\mathcal{B}} \mathbf{b}_0(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x})\, d\mathbf{x} + \int_{\partial\mathcal{B}_t} \mathbf{s}_0(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x})\, d\mathcal{H}^{n-1},$$

in the class of admissible functions

$$\mathcal{A}_m^p := \{\mathbf{u} \in W^{1,p}(\mathcal{B};\mathbb{R}^n) : \mathbf{u} = \mathbf{d} \text{ on } \partial\mathcal{B}_d,\ \mathbf{u} \in \mathrm{Def}^{\,p}(\mathcal{B}, \mathbf{a}_1, ..., \mathbf{a}_M)\}.$$

Here the equality on $\partial\mathcal{B}_d$ is taken to be in the sense of trace.

Due to the lack of an apriori L^∞-bound on such admissible deformations we must strengthen hypothesis (iii) to

(iii)′ (coercivity)

$$W(\mathbf{x}, \mathbf{u}, \mathbf{F}) \geq \phi(\mathbf{x}) + c|\mathbf{F}|^p + \Gamma(\det \mathbf{F})$$

for a.e. $\mathbf{x} \in \mathcal{B}$ and every $(\mathbf{u}, \mathbf{F}) \in \mathbb{R}^3 \times \mathrm{Lin}^3$ with $\det \mathbf{F} > 0$, where $c > 0$, $\phi \in L^1(\mathcal{B})$, and $\Gamma : (0, \infty) \to \mathbb{R}$ is a convex function that satisfies $\Gamma(t)/t \to +\infty$ as $t \to +\infty$.

A slight modification of the proof of Theorem 4.1 then allows us to conclude

Theorem 4.3 *Let $n = 3$, $p > 2$, $\mathbf{d} \in L^p(\partial\mathcal{B}_d;\mathbb{R}^3)$, $\mathbf{b}_0 \in L^r(\mathcal{B};\mathbb{R}^3)$, and $\mathbf{s}_0 \in L^q(\partial\mathcal{B}_t;\mathbb{R}^3)$ be given, where q and r are chosen so that the linear mapping $L : W^{1,p}(\mathcal{B};\mathbb{R}^3) \to \mathbb{R}$ is continuous. Suppose that W satisfies the hypotheses of Theorem 4.1 with (iii) replaced by (iii)′. Assume further that I is given by (4.3), where $\eta_i \geq 0$ for $i = 1, ..., M$ and $\upsilon : [0, \infty)^M \to [0, \infty)$ are lower semicontinuous. Then, if $\mathcal{A}_m^p$ is nonempty, $\hat{E}$ and $\hat{I}$ each attain their infimum on $\mathcal{A}_m^p$.*

Remark. In particular, by the Sobolev imbedding and trace theorems, the linear mapping $L : W^{1,p}(\mathcal{B};\mathbb{R}^3) \to \mathbb{R}$ will be continuous if $r \geq 3p/(4p - 3)$ and $q \geq 2p/3(p - 1)$. See [2] or [8] for the appropriate modification to the proof when such dead-load body forces and surface tractions are included.

5. The Equilibrium Equations

In this section we take $\mathcal{B} \subset \mathbb{R}^n$ and consider equilibrium conditions that are satisfied by minimizers of the energy

$$I(\mathbf{u}) = \int_{\mathcal{B}} W(\nabla\mathbf{u}(\mathbf{x}))\, d\mathbf{x} + \upsilon(\alpha^1, ..., \alpha^M) + \sum_{i=1}^{M} H(\alpha^i)\eta_i.$$

when $W : \mathrm{Lin}^{>} \to \mathbb{R}$ is a smooth function. For simplicity we have restricted our attention to the case where W does not depend on $\mathbf{x}$ or $\mathbf{u}$. If such a minimizer is C^2 it must satisfy the Euler-Lagrange equations

$$\operatorname{div} \mathbf{S}(\nabla \mathbf{u}) \equiv \mathbf{0}, \tag{5.1}$$

where

$$\mathbf{S}(\mathbf{F}) := \frac{dW}{d\mathbf{F}}$$

is the **Piola-Kirchhoff stress**.

If $\mathbf{u}$ is only in $W^{1,p}(\mathcal{B};\mathbb{R}^n)$ then it is not known whether (5.1) holds (in the sense of distributions), even if $p > n$. The difficulty is that to derive (5.1) one usually considers variations $\mathbf{u} + s\mathbf{v}$, $\mathbf{v} \in C_0^\infty(\mathcal{B};\mathbb{R}^n)$. However, when $\mathbf{u}$ is in such a Sobolev space it is not clear that any such variation has finite energy since $\det(\nabla\mathbf{u} + s\nabla\mathbf{v})$ may be negative on a set of positive measure. Ball [4] observed, when $\upsilon = 0$ and $\eta_i = 0$, that one can still derive other equilibrium equations if one instead considers outer variations $\mathbf{g}_s \circ \mathbf{u}$ or inner variations $\mathbf{u} \circ \mathbf{g}_s$ where $\mathbf{g}_s$ is a one-parameter family of diffeomorphisms with $\mathbf{g}_0 = \mathbf{id}$. The former variations lead to conditions in the deformed configuration while the latter give the energy-momentum equations. We begin with outer variations and first fix some notation.

Let $\mathbf{u} \in W^{1,1}(\mathcal{B};\mathbb{R}^n)$ be one-to-one almost everywhere and satisfy $\det \nabla \mathbf{u} > 0$ a.e. Suppose that $N_{\mathbf{u}}$ is the null set given by the change of variables formula (Proposition 2.7), $M_{\mathbf{u}}$ is the null set where $\det \nabla\mathbf{u} = 0$, and $P_{\mathbf{u}}$ is a null set such that $\mathbf{u}|_{\mathcal{B}\setminus P_{\mathbf{u}}}$ is one-to-one. Define $Q_{\mathbf{u}} := M_{\mathbf{u}} \cup N_{\mathbf{u}} \cup P_{\mathbf{u}}$. Then we define the **Cauchy stress** $\mathbf{T} : \mathbf{u}(\mathcal{B}\setminus Q_{\mathbf{u}}) \to \mathrm{Lin}$ by

$$\mathbf{T}(\mathbf{y}) := \mathbf{S}(\nabla\mathbf{u}(\mathbf{x}))\,[\nabla\mathbf{u}(\mathbf{x})]^T / \det \nabla\mathbf{u}(\mathbf{x}), \tag{5.2}$$

or, in components,

$$T^i_j = \sum_{\alpha=1}^{3} \frac{\partial W}{\partial F^i_\alpha} \frac{\partial u^j}{\partial x_\alpha} [\det \nabla\mathbf{u}]^{-1}.$$

Here $\mathbf{x}$ is the unique point in $\mathcal{B}\setminus Q_{\mathbf{u}}$ that satisfies $\mathbf{u}(\mathbf{x}) = \mathbf{y}$.

Theorem 5.1 *Let $\upsilon \equiv 0$. Suppose that W is C^1 on $\mathrm{Lin}^{>}$ and that there are constants $C > 0$ and $\varepsilon > 0$ such that*

$$|\mathbf{S}(\mathbf{A}\mathbf{F})\mathbf{F}^T| \le C(W(\mathbf{F}) + 1) \tag{5.3}$$

for all $\mathbf{F} \in \mathrm{Lin}^{>}$ and all $\mathbf{A} \in \mathrm{Lin}^{>}$ that satisfy $|\mathbf{A} - \mathbf{Id}| < \varepsilon$. If $\mathbf{u} \in \mathcal{A}^p$ is any minimizer of I with finite energy then $\mathbf{T} \in L^1(\mathbf{u}(\mathcal{B}\setminus Q_{\mathbf{u}});\mathrm{Lin})$ and

$$\int_{\mathbf{u}(\mathcal{B}\setminus Q_{\mathbf{u}})} \mathbf{T} \cdot \nabla\mathbf{v}\, d\mathbf{y} = 0 \tag{5.4}$$

for all $\mathbf{v} \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ *that satisfy* $\mathbf{v} = \mathbf{0}$ *on* $\mathbb{R}^n \backslash \mathbf{d}(\mathcal{B})$*. Moreover, if* $\mathbf{u} \in C^2(\mathcal{B}\backslash\{\mathbf{a}_1, ..., \mathbf{a}_M\};\mathbb{R}^n)$ *satisfies* $\det \nabla\mathbf{u} > 0$ *on* $\mathcal{B}\backslash\{\mathbf{a}_1, ..., \mathbf{a}_M\}$ *then*

$$\operatorname{div}_{\mathbf{x}} \mathbf{S}(\nabla \mathbf{u}) = \mathbf{0} \quad \text{in} \quad \mathcal{B}\backslash\{\mathbf{a}_1, ..., \mathbf{a}_M\} \tag{5.5}$$

and if, in addition, $\mathbf{u}$ *is one-to-one on* $\mathcal{B}\backslash\{\mathbf{a}_1, ..., \mathbf{a}_M\}$ *then*

$$\operatorname{div}_{\mathbf{y}} \mathbf{T} = \mathbf{0} \quad \text{in} \quad \mathbf{u}(\mathcal{B}\backslash\{\mathbf{a}_1, ..., \mathbf{a}_M\}). \tag{5.6}$$

Remarks. 1. When $\mathbf{u} \in \mathcal{A}^p_m$ a slight modification of our proof will also yield (5.4)–(5.5).

2. If the energy term that depends on hole volume, v, is not zero then the equilibrium equations will contain an additional term, see, e.g., [23].

Proof. Let $\mathbf{v} \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ satisfy $\mathbf{v} = \mathbf{0}$ on $\mathbb{R}^n\backslash\mathbf{d}(\mathcal{B})$. Then $\mathbf{g}_s := \mathbf{id} + s\mathbf{v}$ is a diffeomorphism of $\mathbb{R}^n$ for small s. Define $\mathbf{u}_s := \mathbf{g}_s \circ \mathbf{u}$ so that, by Corollary 6.4, $\mathbf{u}_s \in \mathcal{A}^p$ for sufficiently small s. With the help of (5.3) one can use the Lebesgue dominated convergence theorem to show that (see [4], [6], and [14]) $\mathbf{T} \in L^1(\mathbf{u}(\mathcal{B}\backslash Q_{\mathbf{u}}); \mathrm{Lin})$ and

$$\begin{aligned}
\frac{d}{ds}\bigg|_{s=0} \int_{\mathcal{B}} W(\nabla\mathbf{u}_s)\, d\mathbf{x} &= \int_{\mathcal{B}} \frac{d}{ds}\bigg|_{s=0} W(\nabla\mathbf{u}_s)\, d\mathbf{x} \\
&= \int_{\mathcal{B}} \mathbf{S}(\nabla\mathbf{u}) \cdot [((\nabla\mathbf{v})\circ\mathbf{u})\,\nabla\mathbf{u}]\, d\mathbf{x} \\
&= \int_{\mathcal{B}} \mathbf{T}(\mathbf{u}(\mathbf{x})) \cdot [(\nabla\mathbf{v})(\mathbf{u}(\mathbf{x}))] \det\nabla\mathbf{u}(\mathbf{x})\, d\mathbf{x}
\end{aligned}$$

and hence, by the change of variables formula (Proposition 2.7),

$$\frac{d}{ds}\bigg|_{s=0} \int_{\mathcal{B}} W(\nabla\mathbf{u}_s)\, d\mathbf{x} = \int_{\mathbf{u}(\mathcal{B}\backslash Q_{\mathbf{u}})} \mathbf{T}(\mathbf{y}) \cdot \nabla\mathbf{v}(\mathbf{y})\, d\mathbf{y}.$$

We note that $\mathbf{g}_s$ is uniformly close to $\mathbf{id}$ and hence that $\mathbf{g}_s$ cannot close (or open) any holes when s is sufficiently small. It follows that the additional cavitation energy does not change. Thus, since $\mathbf{u}$ minimizes I in $\mathcal{A}^p$ we conclude that (5.4) is satisfied.

Suppose now that $\mathbf{u} \in C^2(\mathcal{B}\backslash\{\mathbf{a}_1, ..., \mathbf{a}_M\};\mathbb{R}^n)$ satisfies $\det\nabla\mathbf{u} > 0$ on $\mathcal{B}\backslash\{\mathbf{a}_1, ..., \mathbf{a}_M\}$ and let $\mathbf{b} \in \mathcal{B}\backslash\{\mathbf{a}_1, ..., \mathbf{a}_M\}$. Then since $\det\nabla\mathbf{u}(\mathbf{b}) > 0$ we can apply the inverse function theorem to conclude that, for r sufficiently small, $\mathbf{u}|_{B(\mathbf{b},r)}$ is a diffeomorphism and $\mathbf{u}(B(\mathbf{b}, r))$ is open. Therefore we can define the Cauchy stress $\mathbf{T}$ by (5.2) *on the entire set* $\mathbf{u}(B(\mathbf{b}, r))$.

Now let $\mathbf{v} \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ be supported in $\mathbf{u}(B(\mathbf{b}, r))$. Then by the previous argument we find that

$$0 = \int_{\mathbf{u}(B(\mathbf{b},r))} \mathbf{T} \cdot \nabla\mathbf{v}\, d\mathbf{y}$$

and hence, since $\mathbf{v} \in C^1$ is arbitrary and $\mathbf{T}$ is C^1, a standard result yields

$$\operatorname{div}_{\mathbf{y}} \mathbf{T} = \mathbf{0} \quad \text{in} \quad \mathbf{u}(B(\mathbf{b}, r)). \tag{5.7}$$

Since $\mathbf{u}$ is a diffeomorphism on $B(\mathbf{b}, r)$ we can now apply the identity (see, e.g., Theorem 1.7-1 in [8])

$$\operatorname{div}_{\mathbf{x}} \mathbf{S}(\nabla \mathbf{u}(\mathbf{x})) = [\det \nabla \mathbf{u}(\mathbf{x})] \operatorname{div}_{\mathbf{y}} \mathbf{T}(\mathbf{u}(\mathbf{x}))$$

to conclude

$$\operatorname{div}_{\mathbf{x}} \mathbf{S}(\nabla \mathbf{u}) = \mathbf{0} \quad \text{in} \quad B(\mathbf{b}, r).$$

Therefore (5.5) is satisfied since $\mathbf{b} \in \mathcal{B}\backslash\{\mathbf{a}_1, ..., \mathbf{a}_M\}$ is arbitrary. Finally, if $\mathbf{u}$ is one-to-one the Cauchy stress $\mathbf{T}$ can be defined uniquely on $\mathbf{u}(\mathcal{B}\backslash\{\mathbf{a}_1, ..., \mathbf{a}_M\})$ by (5.2) and (5.6) follows immediately from (5.7). □

We next consider inner variations.

Theorem 5.2 *Let W be C^1 on $\mathrm{Lin}^>$ and suppose that there are constants $C > 0$ and $\varepsilon > 0$ such that*

$$|\mathbf{F}^T \mathbf{S}(\mathbf{F}\mathbf{A})| \leq C(W(\mathbf{F}) + 1) \tag{5.8}$$

for all $\mathbf{F} \in \mathrm{Lin}^>$ and all $\mathbf{A} \in \mathrm{Lin}^>$ that satisfy $|\mathbf{A} - \mathbf{Id}| < \varepsilon$. If $\mathbf{u} \in \mathcal{A}^p$ (or $\mathcal{A}^p_m$) is any minimizer of I with finite energy then

$$\int_{\mathcal{B}} \left[W(\nabla \mathbf{u})\mathbf{Id} - (\nabla \mathbf{u})^T \mathbf{S}(\nabla \mathbf{u})\right] \cdot \nabla \mathbf{v} \, d\mathbf{x} = 0 \tag{5.9}$$

for every $\mathbf{v} \in C^1(\overline{\Omega}; \mathbb{R}^n)$ that satisfies $\mathbf{v} = \mathbf{0}$ on $\{\mathbf{a}_1, ..., \mathbf{a}_M\} \cup (\overline{\Omega} \setminus \mathcal{B})$. Moreover, if $\mathbf{u} \in C^2(\mathcal{B}\backslash\{\mathbf{a}_1, ..., \mathbf{a}_M\}; \mathbb{R}^n)$ then

$$\operatorname{div}\left[W(\nabla \mathbf{u})\mathbf{Id} - (\nabla \mathbf{u})^T \mathbf{S}(\nabla \mathbf{u})\right] = \mathbf{0} \quad \text{in} \quad \mathcal{B}\backslash\{\mathbf{a}_1, ..., \mathbf{a}_M\} \tag{5.10}$$

or, in components,

$$\sum_{\alpha, j} \frac{\partial}{\partial x^\alpha} \left\{ W(\nabla \mathbf{u}) \delta^\alpha_\beta - \frac{\partial W}{\partial F^j_\alpha}(\nabla \mathbf{u}) \frac{\partial u_j}{\partial x_\beta} \right\} = 0.$$

Remark. The above result is *not* valid when W depends on $\mathbf{x}$.

Proof. Let $\mathbf{v} \in C^1(\overline{\Omega}; \mathbb{R}^n)$ satisfy $\mathbf{v} = \mathbf{0}$ on $\{\mathbf{a}_1, ..., \mathbf{a}_M\} \cup (\overline{\Omega} \setminus \mathcal{B})$. Then $\mathbf{h}_s := \mathbf{id} + s\mathbf{v}$ is a diffeomorphism of $\mathcal{B}$ for small s. Let $\mathbf{u} \in \mathcal{A}^p$ ($\mathbf{u} \in \mathcal{A}^p_m$) have finite energy and minimize I. Define $\mathbf{u}_s := \mathbf{u} \circ \mathbf{g}_s$, where $\mathbf{g}_s := (\mathbf{h}_s)^{-1}$. Then, by Corollary 6.6, $\mathbf{u}_s \in \mathcal{A}^p$ ($\mathbf{u}_s \in \mathcal{A}^p_m$) for sufficiently small s.

Define $\mathbf{F}(\mathbf{x}) := \nabla \mathbf{u}(\mathbf{x})$, $\mathbf{G}_s(\mathbf{z}) := \nabla \mathbf{g}_s(\mathbf{z})$, and $\mathbf{H}_s(\mathbf{x}) := \nabla \mathbf{h}_s(\mathbf{x})$. Then if one takes the gradient, with respect to $\mathbf{x}$, of the identity $\mathbf{g}_s(\mathbf{h}_s(\mathbf{x})) = \mathbf{x}$ one concludes that $\mathbf{G}_s(\mathbf{h}_s(\mathbf{x}))\mathbf{H}_s(\mathbf{x}) = \mathbf{Id}$ and hence that

$$\mathbf{G}_s(\mathbf{h}_s(\mathbf{x})) = \mathbf{H}_s(\mathbf{x})^{-1}. \tag{5.11}$$

For future use we note that upon differentiating the identity $\mathbf{H}_s(\mathbf{x})^{-1}\mathbf{H}_s(\mathbf{x}) = \mathbf{Id}$ with respect to s one deduces that

$$\frac{d}{ds}\,\mathbf{H}_s(\mathbf{x})^{-1} = -\mathbf{H}_s(\mathbf{x})^{-1}\left[\frac{d}{ds}\,\mathbf{H}_s(\mathbf{x})\right]\mathbf{H}_s(\mathbf{x})^{-1},$$

and therefore, since $\mathbf{h}_s = \mathbf{id} + s\mathbf{v}$,

$$\frac{d}{ds}\,\mathbf{H}_s(\mathbf{x})^{-1}\bigg|_{s=0} = -\nabla\mathbf{v}. \tag{5.12}$$

Now, by (5.11) and the change of variables formula for diffeomorphisms,

$$\begin{aligned}\int_{\mathcal{B}} W(\nabla\mathbf{u}_s(\mathbf{z}))\,d\mathbf{z} &= \int_{\mathcal{B}} W(\mathbf{F}(\mathbf{g}_s(\mathbf{z}))\mathbf{G}_s(\mathbf{z}))\det\mathbf{G}_s(\mathbf{z})\det\mathbf{H}_s(\mathbf{g}_s(\mathbf{z}))\,d\mathbf{z}\\ &= \int_{\mathcal{B}} W(\mathbf{F}(\mathbf{x})\mathbf{H}_s(\mathbf{x})^{-1})\det\mathbf{H}_s(\mathbf{x})\,d\mathbf{x}\end{aligned}$$

and consequently, with the aid of (5.8), one can use the Lebesgue dominated convergence theorem to show that (see [4], [6], and [14])

$$\begin{aligned}\frac{d}{ds}\bigg|_{s=0}\int_{\mathcal{B}} W(\nabla\mathbf{u}_s(\mathbf{z}))\,d\mathbf{z} &= \frac{d}{ds}\bigg|_{s=0}\int_{\mathcal{B}} W(\mathbf{F}(\mathbf{x})\mathbf{H}_s(\mathbf{x})^{-1})\det\mathbf{H}_s(\mathbf{x})\,d\mathbf{x}\\ &= \int_{\mathcal{B}}\frac{d}{ds}\bigg|_{s=0}\Big[W(\mathbf{F}(\mathbf{x})\mathbf{H}_s(\mathbf{x})^{-1})\det\mathbf{H}_s(\mathbf{x})\Big]\,d\mathbf{x}.\end{aligned} \tag{5.13}$$

We note that

$$\frac{d}{ds}\,\det\mathbf{H}_s(\mathbf{x}) = (\operatorname{adj}\mathbf{H}_s(\mathbf{x}))\cdot\frac{d}{ds}\,\mathbf{H}_s(\mathbf{x})$$

and hence, since $\mathbf{h}_s = \mathbf{id} + s\mathbf{v}$,

$$\frac{d}{ds}\bigg|_{s=0}\det\mathbf{H}_s(\mathbf{x}) = \mathbf{Id}\cdot\nabla\mathbf{v}. \tag{5.14}$$

Thus, by the product rule, (5.12), and (5.14),

$$\frac{d}{ds}\bigg|_{s=0}\Big[W(\mathbf{F}(\mathbf{x})\mathbf{H}_s(\mathbf{x})^{-1})\det\mathbf{H}_s(\mathbf{x})\Big] = \left[W(\nabla\mathbf{u})\mathbf{Id} - (\nabla\mathbf{u})^T\frac{dW}{d\mathbf{F}}\,(\nabla\mathbf{u})\right]\cdot\nabla\mathbf{v},$$

which together with (5.13) yields

$$\frac{d}{ds}\bigg|_{s=0}\int_{\mathcal{B}} W(\nabla\mathbf{u}_s(\mathbf{z}))\,d\mathbf{z} = \int_{\mathcal{B}}\Big[W(\nabla\mathbf{u})\mathbf{Id} - (\nabla\mathbf{u})^T\mathbf{S}(\nabla\mathbf{u})\Big]\cdot\nabla\mathbf{v}\,d\mathbf{x}.$$

In view of Theorem 6.5 the additional energy due to cavitation is invariant under composition on the right with a diffeomorphism. Thus, since $\mathbf{u}$ minimizes I in $\mathcal{A}^p$ (or $\mathcal{A}^p_m$) we conclude that (5.9) is satisfied. Finally, if $\mathbf{u}$ is C^2 away from the cavitation points a standard argument yields (5.10). □

6. Invariance under Composition by Diffeomorphisms

Let $\mathcal{B} \subset \mathbb{R}^n$ and suppose that $\mathbf{g} : \overline{B} \to \mathbb{R}^n$ is an orientation preserving diffeomorphism of $\mathcal{B}$. Then one can view either $\mathcal{B}$ or $\mathbf{g}(\mathcal{B})$ as the reference configuration for the body. Any restrictions that one imposes on deformations, such as $\mathbf{u} \in W^{1,p}$, $\det \nabla \mathbf{u} > 0$, a.e., or condition (INV); or any restrictions that one imposes on the constitutive relation, such as polyconvexity, should not depend on this arbitrary choice of reference configuration. In this section we will show that our class of admissible deformations is invariant under such a change in reference configuration. We will also show that if after deforming the body by an admissible deformation $\mathbf{u}$ one further deforms the body by using a diffeomorphism $\mathbf{g}$ of $\mathbb{R}^n$ then the composition $\mathbf{g} \circ \mathbf{u}$ is also admissible. We first consider the invariance of our class of deformations under composition on the left by a diffeomorphism.

Lemma 6.1 *Let* $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ *be a diffeomorphism that satisfies* $\det \nabla \mathbf{g} > 0$ *on* $\mathbb{R}^n$. *Suppose that* $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^n)$, *with* $p > n-1$, *satisfies* $\det \nabla \mathbf{u} > 0$ *a.e. If* $\mathbf{u}^*$ *satisfies condition (INV) then so does* $\mathbf{g} \circ \mathbf{u}^*$. *Moreover, for every* $\mathbf{b} \in \Omega$,

$$\mathbf{g}\left(\mathrm{im}_T(\mathbf{u}^*, B(\mathbf{b}, r))\right) = \mathrm{im}_T(\mathbf{g} \circ \mathbf{u}^*, B(\mathbf{b}, r)) \tag{6.1}$$

for a.e. every $r \in (0, r_{\mathbf{b}})$.

Proof. As usual we will write $\mathbf{u}$ for $\mathbf{u}^*$. Let $\mathbf{b} \in \Omega$ and $r \in (0, r_{\mathbf{b}})$ be such that $\mathbf{u}|_{\partial B(\mathbf{b},r)}$ is continuous and (i) and (ii) of condition (INV) are satisfied. We first consider (i) of (INV). Let $N \subset B(\mathbf{b}, r)$ be a Lebesgue null set such that

$$\mathbf{u}(\mathbf{x}) \in \mathrm{im}_T(\mathbf{u}, B(\mathbf{b}, r)) \cup \mathbf{u}(\partial B(\mathbf{b}, r)) \quad \text{for every} \quad \mathbf{x} \in B(\mathbf{b}, r) \backslash N.$$

Fix $\mathbf{x} \in B(\mathbf{b}, r) \backslash N$. If $\mathbf{u}(\mathbf{x}) \in \mathbf{u}(\partial B(\mathbf{b}, r))$ then

$$\mathbf{g}(\mathbf{u}(\mathbf{x})) \in \mathbf{g}\left(\mathbf{u}(\partial B(\mathbf{b}, r))\right). \tag{6.2}$$

Otherwise, $\mathbf{u}(\mathbf{x}) \in \mathrm{im}_T(\mathbf{u}, B(\mathbf{b}, r))$ and hence by the definition of the topological image $\deg(\mathbf{u}, \partial B(\mathbf{b}, r), \mathbf{u}(\mathbf{x})) \neq 0$. Thus, by the multiplicative property of degree (Proposition 2.1(ii)) $\deg(\mathbf{u}, \partial B(\mathbf{b}, r), \mathbf{g}(\mathbf{u}(\mathbf{x}))) \neq 0$ and consequently, by the definition of the topological image,

$$\mathbf{g}(\mathbf{u}(\mathbf{x})) \in \mathrm{im}_T(\mathbf{g} \circ \mathbf{u}, B(\mathbf{b}, r)). \tag{6.3}$$

Therefore, by (6.2) and (6.3),

$$\mathbf{g}(\mathbf{u}(\mathbf{x})) \in \mathrm{im}_T(\mathbf{g}\circ\mathbf{u}, B(\mathbf{b},r)) \cup (\mathbf{g}\circ\mathbf{u})(\partial B(\mathbf{b},r)) \quad \text{for every} \quad \mathbf{x} \in B(\mathbf{b},r)\backslash N,$$

which shows that $\mathbf{g}\circ\mathbf{u}$ satisfies (i) of (INV). The proof of (ii) of (INV) is similar.

We now consider (6.1). By the multiplicative property of degree, Proposition 2.1(ii),

$$\deg(\mathbf{g}\circ\mathbf{u}, \partial B(\mathbf{b},r), \mathbf{g}(\mathbf{y})) = \deg(\mathbf{u}, \partial B(\mathbf{b},r), \mathbf{y})$$

for every $\mathbf{y} \in \mathbb{R}^n\backslash\mathbf{u}(\partial B(\mathbf{b},r))$. Thus the definition of the topological image yields

$$\mathbf{g}(\mathbf{y}) \in \mathrm{im}_T(\mathbf{g}\circ\mathbf{u}, B(\mathbf{b},r)) \Longleftrightarrow \mathbf{y} \in \mathrm{im}_T(\mathbf{u}, B(\mathbf{b},r)).$$

Equation (6.1) now follows. □

Lemma 6.2 *Let* $\mathbf{g}$ *and* $\mathbf{u}$ *satisfy the hypotheses of Lemma 6.1. Suppose, in addition, that* $\det\nabla\mathbf{g} \in L^\infty(\mathbb{R}^n)$*. Then*

$$(\mathrm{Det}\,\nabla(\mathbf{g}\circ\mathbf{u}))(A) \le ||\det\nabla\mathbf{g}||_\infty(\mathrm{Det}\,\nabla\mathbf{u})(A), \tag{6.4}$$

for every $A \subset \Omega$ *and hence the Radon measure* $\mathrm{Det}\,\nabla(\mathbf{g}\circ\mathbf{u})$ *is absolutely continuous with respect to* $\mathrm{Det}\,\nabla\mathbf{u}$.

Proof. As usual we will write $\mathbf{u}$ for $\mathbf{u}^*$. Fix $\mathbf{b} \in \Omega$. Then by Lemma 6.1 and Proposition 2.9 we have that for $\mathcal{L}^1$ a.e. $r \in (0, r_{\mathbf{b}})$

$$(\mathrm{Det}\,\nabla\mathbf{u})(\overline{B(\mathbf{b},r)}) = \mathcal{L}^n(\mathrm{im}_T(\mathbf{u}, B(\mathbf{b},r))) \tag{6.5}$$

and

$$(\mathrm{Det}\,\nabla(\mathbf{g}\circ\mathbf{u}))(\overline{B(\mathbf{b},r)}) = \mathcal{L}^n(\mathrm{im}_T(\mathbf{g}\circ\mathbf{u}, B(\mathbf{b},r))).$$

Thus, in view of (6.1),

$$(\mathrm{Det}\,\nabla(\mathbf{g}\circ\mathbf{u}))(\overline{B(\mathbf{b},r)}) = \mathcal{L}^n\left(\mathbf{g}\left(\mathrm{im}_T(\mathbf{u}, B(\mathbf{b},r))\right)\right). \tag{6.6}$$

However, by the change of variables formula for diffeomorphisms (see Proposition 2.7 and the remark that follows it), for any Lebesgue measurable set $A \subset \Omega$,

$$\mathcal{L}^n(\mathbf{g}(A)) = \int_A \det\nabla\mathbf{g}(\mathbf{x})\,d\mathbf{x} \le ||\det\nabla\mathbf{g}||_\infty\mathcal{L}^n(A). \tag{6.7}$$

Therefore, if we take $A = \overline{B(\mathbf{b},r)}$ in (6.7) we find, with the aid of (6.5) and (6.6), that

$$(\mathrm{Det}\,\nabla(\mathbf{g}\circ\mathbf{u}))(\overline{B(\mathbf{b},r)}) \le ||\det\nabla\mathbf{g}||_\infty(\mathrm{Det}\,\nabla\mathbf{u})(\overline{B(\mathbf{b},r)}).$$

A standard result on Radon measures (see, e.g., Lemma 1(i) in section 1.6.1 in [10]) then yields (6.4). □

Theorem 6.3 *Let* $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ *be a diffeomorphism that satisfies* $\det \nabla \mathbf{g} \in L^\infty(\mathbb{R}^n)$ *and* $\det \nabla \mathbf{g} > 0$ *on* $\mathbb{R}^n$. *Suppose that*

$$\mathbf{u} \in \mathrm{Def}^{\,p}(\Omega, \mathbf{a}_1, ..., \mathbf{a}_M).$$

Then

$$\mathbf{g} \circ \mathbf{u} \in \mathrm{Def}^{\,p}(\Omega, \mathbf{a}_1, ..., \mathbf{a}_M).$$

Proof. Let $\mathbf{u} \in \mathrm{Def}^{\,p}(\Omega, \mathbf{a}_1, ..., \mathbf{a}_M)$ so that $\mathbf{u}^*$ satisfies condition (INV), $\det \nabla \mathbf{u} > 0$ a.e., and

$$\mathrm{Det}\, \nabla \mathbf{u} = (\det \nabla \mathbf{u})\mathcal{L}^n + \sum_{i=1}^{M} \alpha^i \delta_{\mathbf{a}_i}. \tag{6.8}$$

Let $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism that satisfies $\det \nabla \mathbf{g} \in L^\infty(\mathbb{R}^n)$ and $\det \nabla \mathbf{g} > 0$ on $\mathbb{R}^n$. Then, by Lemma 6.1, $\mathbf{g} \circ \mathbf{u}^*$ satisfies condition (INV). Since $\det \nabla(\mathbf{g} \circ \mathbf{u}) = (\det \nabla \mathbf{g})(\det \nabla \mathbf{u})$ it is clear that $\det \nabla(\mathbf{g} \circ \mathbf{u}) > 0$ a.e. Thus the desired result will follow once we show that there exist $\beta^i \geq 0$ such that

$$\mathrm{Det}\, \nabla(\mathbf{g} \circ \mathbf{u}) = (\det \nabla(\mathbf{g} \circ \mathbf{u}))\mathcal{L}^n + \sum_{i=1}^{M} \beta^i \delta_{\mathbf{a}_i}. \tag{6.9}$$

By Proposition 2.9 there is a Radon measure m, which is singular with respect to Lebesgue measure, such that

$$\mathrm{Det}\, \nabla(\mathbf{g} \circ \mathbf{u}) = (\det \nabla(\mathbf{g} \circ \mathbf{u}))\mathcal{L}^n + m. \tag{6.10}$$

Let $N \subset \Omega$ be a Lebesgue null set. Define $N^- := N \backslash \{\mathbf{a}_1, ..., \mathbf{a}_M\}$. Then, by (6.8), N^- is a $\mathrm{Det}\, \nabla \mathbf{u}$ null set and hence, by Lemma 6.2, N^- is a $\mathrm{Det}\, \nabla(\mathbf{g} \circ \mathbf{u})$ null set. Therefore, $m(N \backslash \{\mathbf{a}_1, ..., \mathbf{a}_M\}) = 0$ for every Lebesgue null set N. Since m is singular with respect to Lebesgue measure it follows that there exist $\beta^i \geq 0$ such that

$$m = \sum_{i=1}^{M} \beta^i \delta_{\mathbf{a}_i},$$

which together with (6.10) yields (6.9). □

Corollary 6.4 *Let* $\mathbf{u} \in \mathcal{A}^p$. *Suppose that* $\mathbf{v} \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ *satisfies* $\mathbf{v} = \mathbf{0}$ *on* $\mathbb{R}^n \backslash \mathbf{d}(\mathcal{B})$. *Define*

$$\mathbf{g}_s := \mathbf{id} + s\mathbf{v} \quad \textit{and} \quad \mathbf{u}_s := \mathbf{g}_s \circ \mathbf{u}.$$

Then, for all sufficiently small s,

$$\mathbf{u}_s \in \mathcal{A}^p.$$

Remark. A similar result is valid for $\mathbf{u} \in \mathcal{A}^p_m$.

Proof. Let $\mathbf{v} \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ satisfy $\mathbf{v} = \mathbf{0}$ on $\mathbb{R}^n \backslash \mathbf{d}(\mathcal{B})$. Then it follows that (see, e.g., Section 5.5 in [8]), for sufficiently small s, the mapping $\mathbf{g}_s := \mathbf{id} + s\mathbf{v}$ is a diffeomorphism of $\mathbb{R}^n$, $\det \nabla \mathbf{g}_s \in L^\infty(\mathbb{R}^n)$, and $\det \nabla \mathbf{g}_s > 0$ on $\mathbb{R}^n$.

Let $\mathbf{u} \in \mathcal{A}^p$ and define $\mathbf{u}_s := \mathbf{g}_s \circ \mathbf{u}$. Then $\mathbf{u}^e \in \mathrm{Def}^{\,p}(\Omega, \mathbf{a}_1, ..., \mathbf{a}_M)$ and we may therefore apply Theorem 6.3 to conclude

$$(\mathbf{u}^e)_s := \mathbf{g}_s \circ \mathbf{u}^e \in \mathrm{Def}^{\,p}(\Omega, \mathbf{a}_1, ..., \mathbf{a}_M).$$

Next, we note that $\mathbf{v} = \mathbf{0}$ on $\mathbf{d}(\overline{\Omega} \backslash \mathcal{B}) = \mathbf{d}(\overline{\Omega}) \backslash \mathbf{d}(\mathcal{B})$ and consequently $(\mathbf{u}^e)_s(\mathbf{x}) = \mathbf{id}(\mathbf{u}^e(\mathbf{x})) = \mathbf{d}(\mathbf{x})$ for $\mathbf{x} \in \overline{\Omega} \backslash \mathcal{B}$. Therefore, $(\mathbf{u}_s)^e = (\mathbf{u}^e)_s \in \mathrm{Def}^{\,p}(\Omega, \mathbf{a}_1, ..., \mathbf{a}_M)$ and $\mathbf{u} - \mathbf{d} \in W^{1,p}_0(\mathcal{B}; \mathbb{R}^n)$, which completes the proof. □

We now consider the invariance of the class of deformations under composition on the right by a diffeomorphism.

Theorem 6.5 *Let* $\mathbf{g} : \overline{\Omega} \to \overline{\Omega}$ *be a diffeomorphism that satisfies* $\det \nabla \mathbf{g} > 0$ *on* Ω. *Suppose that*

$$\mathbf{u} \in \mathrm{Def}^{\,p}(\Omega, \mathbf{a}_1, ..., \mathbf{a}_M)$$

with

$$\mathrm{Det}\, \nabla \mathbf{u} = (\det \nabla \mathbf{u})\mathcal{L}^n + \sum_{i=1}^{M} \alpha^i \delta_{\mathbf{a}_i}. \tag{6.11}$$

Then

$$\mathbf{u} \circ \mathbf{g} \in \mathrm{Def}^{\,p}(\Omega, \mathbf{g}^{-1}(\mathbf{a}_1), ..., \mathbf{g}^{-1}(\mathbf{a}_M))$$

and

$$\mathrm{Det}\, \nabla(\mathbf{u} \circ \mathbf{g}) = (\det \nabla(\mathbf{u} \circ \mathbf{g}))\mathcal{L}^n + \sum_{i=1}^{M} \alpha^i \delta_{\mathbf{g}^{-1}(\mathbf{a}_i)}. \tag{6.12}$$

Proof. Let $\mathbf{u} \in \mathrm{Def}^{\,p}(\Omega, \mathbf{a}_1, ..., \mathbf{a}_M)$ so that $\mathbf{u}^*$ satisfies condition (INV), $\det \nabla \mathbf{u} > 0$ a.e., and (6.11) is satisfied. Let $\mathbf{g} : \overline{\Omega} \to \overline{\Omega}$ be a diffeomorphism that satisfies $\det \nabla \mathbf{g} > 0$ on Ω. Then, by Theorem 9.1 in [23], $\mathbf{u}^* \circ \mathbf{g}$ satisfies condition (INV). Since $\det \nabla(\mathbf{u} \circ \mathbf{g}) = (\det \nabla \mathbf{u})(\det \nabla \mathbf{g})$ it is clear that $\det \nabla(\mathbf{u} \circ \mathbf{g}) > 0$ a.e. Thus the desired result will follow once we show that (6.12) is satisfied.

Let $\phi \in C_0^1(\Omega)$. Then, by the definition of the distributional Jacobian, the change of variables formula for diffeomorphisms (see Proposition 2.7 and the remark that follows it), and the fact that $\mathbf{g}(\Omega) = \Omega$,

$$\begin{aligned}
(\mathrm{Det}\,\nabla(\mathbf{u}\circ\mathbf{g}))(\phi) &= -\frac{1}{n}\int_\Omega \nabla_{\mathbf{z}}\phi\cdot[\mathrm{adj}\,\nabla_{\mathbf{z}}(\mathbf{u}\circ\mathbf{g})](\mathbf{u}\circ\mathbf{g})\,d\mathbf{z}\\
&= -\frac{1}{n}\int_\Omega \mathbf{G}^T\nabla_{\mathbf{x}}(\phi\circ\mathbf{g}^{-1})(\mathbf{g}(\mathbf{z}))\cdot[\mathrm{adj}\,\mathbf{G}][\mathrm{adj}\,\nabla_{\mathbf{x}}\mathbf{u}(\mathbf{g}(\mathbf{z}))]\mathbf{u}(\mathbf{g}(\mathbf{z}))\,d\mathbf{z}\\
&= -\frac{1}{n}\int_\Omega \nabla_{\mathbf{x}}(\phi\circ\mathbf{g}^{-1})(\mathbf{g}(\mathbf{z}))\cdot[\mathrm{adj}\,\nabla_{\mathbf{x}}\mathbf{u}(\mathbf{g}(\mathbf{z}))]\mathbf{u}(\mathbf{g}(\mathbf{z}))\det\mathbf{G}\,d\mathbf{z}\\
&= -\frac{1}{n}\int_\Omega \nabla(\phi\circ\mathbf{g}^{-1})(\mathbf{x})\cdot[\mathrm{adj}\,\nabla\mathbf{u}(\mathbf{x})]\mathbf{u}(\mathbf{x})\,d\mathbf{x}\\
&= (\mathrm{Det}\,\nabla\mathbf{u})(\phi\circ\mathbf{g}^{-1}),
\end{aligned}$$

where $\mathbf{G} := \nabla\mathbf{g}(\mathbf{z})$ and we have made use of the identity $\mathbf{G}[\mathrm{adj}\,\mathbf{G}] = (\det\mathbf{G})\mathbf{Id}$. Consequently,

$$(\mathrm{Det}\,\nabla(\mathbf{u}\circ\mathbf{g}))(A) = (\mathrm{Det}\,\nabla\mathbf{u})(\mathbf{g}^{-1}(A)),$$

for every measurable set A, which together with (6.11) yields

$$(\mathrm{Det}\,\nabla(\mathbf{u}\circ\mathbf{g}))(A) = \int_{\mathbf{g}^{-1}(A)} \det\nabla\mathbf{u}\,d\mathbf{x} + \Big(\sum_{i=1}^M \alpha^i\delta_{\mathbf{g}^{-1}(\mathbf{a}_i)}\Big)(A).$$

Equation (6.12) then follows from the change of variables formula. □

Corollary 6.6 *Let* $\mathbf{u}\in\mathcal{A}^p$ *(*$\mathbf{u}\in\mathcal{A}^p_m$*). Suppose that* $\mathbf{v}\in C^1(\overline{\Omega};\mathbb{R}^n)$ *satisfies* $\mathbf{v}=\mathbf{0}$ *on* $\{\mathbf{a}_1,...,\mathbf{a}_M\}\cup(\overline{\Omega}\setminus\mathcal{B})$*. Define*

$$\mathbf{h}_s := \mathbf{id} + s\mathbf{v} \quad \textit{and} \quad \mathbf{u}_s := \mathbf{u}\circ(\mathbf{h}_s)^{-1}.$$

Then, for all sufficiently small s*,*

$$\mathbf{u}_s\in\mathcal{A}^p \quad (\mathbf{u}_s\in\mathcal{A}^p_m).$$

Proof. Let $\mathbf{v}\in C^1(\overline{\Omega};\mathbb{R}^n)$ satisfy $\mathbf{v}=\mathbf{0}$ on $\{\mathbf{a}_1,...,\mathbf{a}_M\}\cup(\overline{\Omega}\setminus\mathcal{B})$. Then it follows that (see, e.g., Section 5.5 in [8]), for sufficiently small s, the mapping $\mathbf{h}_s := \mathbf{id}+s\mathbf{v}$ is a diffeomorphism, $\det\nabla\mathbf{h}_s > 0$ on Ω, $\mathbf{h}_s(\Omega)=\Omega$, and $\mathbf{h}_s(\mathbf{x})=\mathbf{x}$ for $\mathbf{x}\in\overline{\Omega}\backslash\mathcal{B}$. Clearly, $\mathbf{h}_s(\mathbf{a}_i)=\mathbf{a}_i$.

Let $\mathbf{u}\in\mathcal{A}^p$ and define $\mathbf{u}_s := \mathbf{u}\circ(\mathbf{h}_s)^{-1}$. Then $\mathbf{u}^e\in\mathrm{Def}^p(\Omega,\mathbf{a}_1,...,\mathbf{a}_M)$ and we may therefore apply Theorem 6.5 to conclude

$$(\mathbf{u}^e)_s := \mathbf{u}^e\circ(\mathbf{h}_s)^{-1}\in\mathrm{Def}^p(\Omega,\mathbf{a}_1,...,\mathbf{a}_M).$$

Next, we note that $\mathbf{v}=\mathbf{0}$ on $\overline{\Omega}\backslash\mathcal{B}$ and hence $(\mathbf{u}^e)_s(\mathbf{x}) = \mathbf{u}^e(\mathbf{id}(\mathbf{x})) = \mathbf{d}(\mathbf{x})$ for $\mathbf{x}\in\overline{\Omega}\backslash\mathcal{B}$. Therefore, $(\mathbf{u}_s)^e = (\mathbf{u}^e)_s\in\mathrm{Def}^p(\Omega,\mathbf{a}_1,...,\mathbf{a}_M)$ and $\mathbf{u}-\mathbf{d}\in W_0^{1,p}(\mathcal{B};\mathbb{R}^n)$, which completes the proof when $\mathbf{u}\in\mathcal{A}^p$. The proof for $\mathbf{u}\in\mathcal{A}^p_m$ is similar. □

A. Appendix

The following result concerns the weak convergence of a sequence of products. Since we could not find it in a standard reference, we include a proof for the convenience of the reader.

Lemma A.1 *Let $\psi_j \in L^\infty(\Omega)$ and $\theta_j \in L^1(\Omega)$ satisfy*

$$\begin{aligned} \psi_j &\to \psi \quad \textit{pointwise a.e.}, \\ \theta_j &\rightharpoonup \theta \quad \textit{in} \quad L^1(\Omega), \end{aligned}$$

where $||\psi_j||_\infty \le K$ for some $K > 0$, $\theta \in L^1(\Omega)$, and $\psi \in L^\infty(\Omega)$. Then

$$\theta_j\psi_j \rightharpoonup \theta\psi \quad \textit{in} \quad L^1(\Omega).$$

Proof. Fix $\varphi \in L^\infty(\Omega)$. Without loss of generality let $K = ||\psi||_\infty = ||\varphi||_\infty$. With a view toward applying Egoroff's theorem let $E \subset \Omega$ be any measurable set that satisfies $\psi_j \to \psi$ uniformly on $\Omega\backslash E$. Then

$$\int_\Omega \varphi(\theta_j\psi_j - \theta\psi)\,d\mathbf{x} = \int_\Omega \varphi(\theta_j - \theta)\psi\,d\mathbf{x} + \int_{\Omega\backslash E} \varphi(\psi_j - \psi)\theta_j\,d\mathbf{x} + \int_E \varphi(\psi_j - \psi)\theta_j\,d\mathbf{x}. \quad \text{(A.1)}$$

The first integral in the right-hand side of (A.1) goes to zero, as $j \to \infty$, by the weak convergence hypothesis. The absolute value of the second integral in the right-hand side of (A.1) is bounded above by

$$K \sup_{\mathbf{x}\in\Omega\backslash E} |\psi_j(\mathbf{x}) - \psi(\mathbf{x})| \int_\Omega |\theta_j|\,d\mathbf{x}.$$

We note that the sequence θ_j converges weakly in $L^1(\Omega)$ and hence the L^1 norm of θ_j is bounded uniformly in j. Thus, this term goes to zero, as $j \to \infty$, since $\psi_j \to \psi$ uniformly on $\Omega\backslash E$.

The absolute value of the last integral in the right-hand side of (A.1) is bounded above by

$$(||\psi_j||_\infty + ||\psi||_\infty)||\varphi||_\infty \int_E |\theta_j|\,d\mathbf{x} \le 2K^2 \int_E |\theta_j|\,d\mathbf{x}. \quad \text{(A.2)}$$

We note that the Dunford-Pettis compactness criterion (see, e.g., Corollary 4.8.11 in [9]) implies that the sequence θ_j is uniformly integrable, that is, the integrals in the right-hand side of (A.2) will go to zero, uniformly in j, as the measure of E is made arbitrarily small. Finally, Egoroff's theorem yields E with arbitrarily small measure. □

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