Chapter 2

Vector Calculus

2.1 Directional Derivatives and Gradients

[Bourne, pp. 97–104] & [Anton, pp. 974–991]

Definition 2.1. Let $f : \Omega \to \mathbb{R}$ be a continuously differentiable scalar field on a region $\Omega \subset \mathbb{R}^3$. Then

$$\operatorname{grad} f \;\equiv\; \nabla f \;:=\; \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k} \tag{2.1}$$

is the **gradient** of f on Ω, which is itself a vector field on Ω.

Definition 2.2. Let $f : \Omega \to \mathbb{R}$ be a continuously differentiable scalar field on $\Omega \subset \mathbb{R}^3$ and let $\hat{\boldsymbol{a}}$ be a unit vector in $\mathbb{R}^3$. Then

$$D_{\hat{\boldsymbol{a}}} f(\boldsymbol{x}_0) \;:=\; \lim_{h \to 0} \frac{f(\boldsymbol{x}_0 + h\hat{\boldsymbol{a}}) - f(\boldsymbol{x}_0)}{h} \tag{2.2}$$

is the **directional derivative** of f in the direction $\hat{\boldsymbol{a}}$ at $\boldsymbol{x}_0 \in \Omega$, i.e. the rate of change of f in the direction of $\hat{\boldsymbol{a}}$. Moreover, for any $\boldsymbol{a} \in \mathbb{R}^3$ we define $D_{\boldsymbol{a}} f := D_{\hat{\boldsymbol{a}}} f$ where $\hat{\boldsymbol{a}} = \boldsymbol{a}/|\boldsymbol{a}|$.

Proposition 2.3. *Let* $\boldsymbol{a} \in \mathbb{R}^3$*. Then*

$$D_{\boldsymbol{a}} f \;=\; \nabla f \cdot \frac{\boldsymbol{a}}{|\boldsymbol{a}|} \;. \tag{2.3}$$

Proof.

(2.4)

(2.5)

□

Example 2.4. Find $D_{\boldsymbol{a}} f(\boldsymbol{x}_0)$ for $f(\boldsymbol{x}) := 2x^2+3y^2+z^2$, $\boldsymbol{a} := (1, 0, -2)^T$, and $\boldsymbol{x}_0 := (2, 1, 3)^T$.

Step 1: Find ∇f:

Step 2: Normalise $\boldsymbol{a}$:

Step 3: Evaluate (2.3) at $\boldsymbol{x}_0$:

Proposition 2.5.

$$\max_{\boldsymbol{a} \in \mathbb{R}^3} |D_{\boldsymbol{a}} f| = |\nabla f| \tag{2.6}$$

and it is attained in the direction $\boldsymbol{a} = \nabla f$.

Proof.

□

Remark 2.6. *(geometric interpretation of* grad*).*

(a) Proposition 2.5 shows that $\nabla f(\boldsymbol{x}_0)$ gives the **direction** and **magnitude** of the largest directional derivative of f at $\boldsymbol{x}_0$, i.e. the largest rate of change of f. For a picture ilustrating this in 2D see [Anton, Figure 14.6.5, p. 978].

(b) A point $\boldsymbol{x}_0 \in \Omega$ where $\nabla f(\boldsymbol{x}_0) = \mathbf{0}$ is called a **stationary point** (since Proposition 2.5 shows that $D_{\boldsymbol{a}} f(\boldsymbol{x}_0) = 0$ for all $\boldsymbol{a} \in \mathbb{R}^3$).

For work integrals the **Fundamental Theorem of Calculus** takes the following form.

Theorem 2.7. *Let* $\phi : \Omega \to \mathbb{R}$ *be a continuously differentiable scalar field, and let* $\mathcal{C}$ *be a curve in* Ω *from* $\boldsymbol{x}_0$ *to* $\boldsymbol{x}_e$. *Then*

$$\int_{\mathcal{C}} \nabla \phi \cdot d\boldsymbol{r} = \phi(\boldsymbol{x}_e) - \phi(\boldsymbol{x}_0) \ . \tag{2.7}$$

Proof. [Problem Sheet 2, Question 5]. □

2.1.1 Application: Level Surfaces and Grad

DIY	Revise until next week *Level Surfaces* from MA10005 (needed on Problem Sheet 4). See also [Anton, pp. 987–989].

Note. Please do not confuse the definition of a **level surface** (as done in MA10005 or [Anton]) with our definition of a (parametric) surface in Equation (1.13).

2.2 Divergence and Curl; the ∇–Operator

[Bourne, pp. 104–118] & [Anton, pp. 1095–1100]

Let us consider ∇f again, but now think as ∇ as an **operator** acting on a scalar field f. Formally we could write

$$\nabla \; = \; \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} \; = \; \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} . \tag{2.8}$$

This is called the **del-operator** or **nabla**, and applying it to a scalar field f we get

$$\nabla f \; = \; \boldsymbol{i}\frac{\partial f}{\partial x} + \boldsymbol{j}\frac{\partial f}{\partial y} + \boldsymbol{k}\frac{\partial f}{\partial z} \; = \; \operatorname{grad} f \; .$$

Now let us apply ∇ to vector fields. Let $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ be a vector field on a region $\Omega \subset \mathbb{R}^3$, where $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$. (As for ordinary vectors $\boldsymbol{ab}$ does not make sense, but we can form the dot and the cross product, i.e. $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{a} \wedge \boldsymbol{b}$.)

Definition 2.8. We define the **divergence of $\boldsymbol{F}$** to be the scalar field

$$\operatorname{div} \boldsymbol{F} \; := \; \nabla \cdot \boldsymbol{F} \; = \; \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \tag{2.9}$$

and the **curl of $\boldsymbol{F}$** to be the vector field

$$\operatorname{curl} \boldsymbol{F} \; := \; \nabla \wedge \boldsymbol{F} \; = \; \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k} \; . \tag{2.10}$$

Example 2.9. Find div and curl of $\boldsymbol{F}(\boldsymbol{x}) = (-y, x, z)^T$.

Remark 2.10. In manipulating with the ∇–operator many rules from ordinary vector algebra apply, **but not all** (don't use vector algebra to prove them!). In particular, the order is very important, i.e. in general

$$\nabla \cdot (g\boldsymbol{F}) \; \neq \; g\nabla \cdot \boldsymbol{F} \; \neq \; \boldsymbol{F} \cdot \nabla g.$$

Applications. In many fields of mathematical physics, e.g. fluid flow, electromagnetic field propagation, etc...

2.2.1 Second order derivatives – the Laplace Operator

Applying the ∇–operator twice gives **five** possible second derivatives:

$$\boxed{\mathbf{A}} \qquad \operatorname{div}(\operatorname{grad} f) \;=\; \nabla \cdot (\nabla f) \;=\; \nabla^2 f \tag{2.11}$$

$$\boxed{\mathbf{B}} \qquad \operatorname{curl}(\operatorname{grad} f) \;=\; \nabla \wedge (\nabla f)$$

$$\boxed{\mathbf{C}} \qquad \operatorname{div}(\operatorname{curl} \boldsymbol{F}) \;=\; \nabla \cdot (\nabla \wedge \boldsymbol{F})$$

$$\boxed{\mathbf{D}} \qquad \operatorname{curl}(\operatorname{curl} \boldsymbol{F}) \;=\; \nabla \wedge (\nabla \wedge \boldsymbol{F})$$

$$\boxed{\mathbf{E}} \qquad \operatorname{grad}(\operatorname{div} \boldsymbol{F}) \;=\; \nabla (\nabla \cdot \boldsymbol{F})$$

Note. Quantities such as $\operatorname{grad}(\operatorname{curl}\boldsymbol{F})$ or $\operatorname{curl}(\operatorname{div}\boldsymbol{F})$ are meaningless.

Proposition 2.11.

(a) $\boxed{\mathbf{B}}$:
$$\operatorname{curl}(\operatorname{grad} f) \;=\; \mathbf{0}, \tag{2.12}$$

(b) $\boxed{\mathbf{C}}$:
$$\operatorname{div}(\operatorname{curl} \boldsymbol{F}) \;=\; 0, \tag{2.13}$$

(c) $\boxed{\mathbf{E}}$ - $\boxed{\mathbf{D}}$:
$$\operatorname{grad}(\operatorname{div} \boldsymbol{F}) \;-\; \operatorname{curl}(\operatorname{curl} \boldsymbol{F}) \;=\; \nabla^2 \boldsymbol{F}. \tag{2.14}$$

Proof. (a)

(b)

(c) [Problem Sheet 5, Question 1(vi)].

□

The operator

$$\nabla^2 \;=\; \nabla \cdot \nabla \;=\; \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2.15}$$

in (2.11) and (2.14) is called the **Laplace–operator** (also denoted Δ). It is very important in mathematical physics. Many of the basic partial differential equations (PDEs) of mathematical physics involve it, e.g. the **Laplace Equation**

$$\nabla^2 \phi \;=\; \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \;=\; 0\ . \tag{2.16}$$

We will come back to this equation in the second part of this course.

2.2.2 Application: Potential Theory [Bourne, pp. 225–243]

Definition 2.12. A vector field $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ is called **irrotational**, if

$$\mathrm{curl}\boldsymbol{F} = \mathbf{0}.$$

In view of Equation (2.12) we can state the following

Proposition 2.13. *Let $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ be a vector field. The following three statements are equivalent:*

(a) $\boldsymbol{F}$ is irrotational.

(b) There exists a scalar field $\phi : \Omega \to \mathbb{R}$ such that $\boldsymbol{F} = \nabla\phi$. ϕ is called the **scalar potential** *of $\boldsymbol{F}$.*

(c) $\boldsymbol{F}$ is conservative (i.e. work integral independent of path).

Proof. [Problem Sheets 2,4 & 8]. □

Remark 2.14. *(Calculating scalar potentials).* Let $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ be a conservative vector field and let $\mathcal{C}$ be an arbitrary curve from $\boldsymbol{a}$ to $\boldsymbol{x}$. Then

$$\phi(\boldsymbol{x}) = \int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} \tag{2.17}$$

is a scalar potential for $\boldsymbol{F}$, if $\boldsymbol{a}$ is **not** a pole of ϕ. See [Problem Sheet 4, Question 4].

Definition 2.15. A vector field $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ is called **solenoidal** (or **divergence-free**) if

$$\mathrm{div}\boldsymbol{F} = 0.$$

Example 2.16. (Application in Electrostatics). Given an electric field $\boldsymbol{E}$ and no sources, the **Maxwell's Equations** state:

Remark 2.17. Let Ω be bounded and simply connected [Bourne, pp. 225-226]. As a consequence of Equation (2.13) we have also **(without proof)**:

(a) A vector field $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ is solenoidal iff there exists a vector field $\boldsymbol{\Psi}$ such that $\boldsymbol{F} = \mathrm{curl}\boldsymbol{\Psi}$. $\boldsymbol{\Psi}$ is called a **vector potential** of $\boldsymbol{F}$ [Bourne, pp. 230–232].

(b) For every vector field $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ there exist a scalar field ϕ and a vector field $\boldsymbol{\Psi}$ such that

$$\boldsymbol{F} = \mathrm{grad}\,\phi + \mathrm{curl}\,\boldsymbol{\Psi}\,, \tag{2.18}$$

i.e. any vector field can be resolved into the sum of an irrotational and a solenoidal part. This is the famous **Helmholtz Theorem** [Bourne, pp. 238–239].

2.3 Differentiation in Curvilinear Coordinates

[Bourne, pp. 118–136]

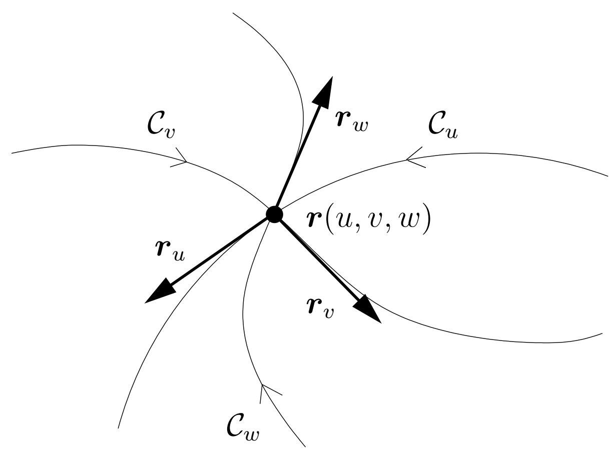
Motivation. So far, in this chapter, we have only looked at scalar fields and vector fields in Cartesian coordinates (x, y, z). What if we have curvilinear coordinates (u, v, w) defined by

$$\boldsymbol{r}(u, v, w) \;=\; x(u, v, w)\,\boldsymbol{i} + y(u, v, w)\,\boldsymbol{j} + z(u, v, w)\,\boldsymbol{k}, \tag{2.19}$$

e.g. spherical polar coordinates, etc. ?

Note. In this section we will always use the notation $x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$ for vectors in Cartesian coordinates rather than $(x, y, z)^T$. I will mention later on why.

We have already seen in Chapter 1 that varying u with v and w fixed creates a curve $\mathcal{C}_u$ in $\mathbb{R}^3$ with tangent vector $\boldsymbol{r}_u := \partial\boldsymbol{r}/\partial u$. Similarly, varying v creates a curve $\mathcal{C}_v$ with tangent vector $\boldsymbol{r}_v := \partial\boldsymbol{r}/\partial v$, and varying w creates a curve $\mathcal{C}_w$ with tangent vector $\boldsymbol{r}_w := \partial\boldsymbol{r}/\partial w$:



2.3.1 Orthogonal Curvilinear Coordinates

Definition 2.18. A triple $(u, v, w) \in D$ together with a Cartesian map $\boldsymbol{r} : D \to \Omega \subset \mathbb{R}^3$ as defined in (2.19) is called a set of **orthogonal curvilinear coordinates** (OCCs) on Ω, if

(a) $\boldsymbol{r}$ is a continuously differentiable bijection with continuously differentiable inverse $\boldsymbol{r}^{-1}$ *almost everywhere*.

(b) The vectors $\boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{r}_w$ are mutually orthogonal, i.e.

$$\boldsymbol{r}_u \cdot \boldsymbol{r}_v \;=\; \boldsymbol{r}_u \cdot \boldsymbol{r}_w \;=\; \boldsymbol{r}_v \cdot \boldsymbol{r}_w = 0.$$

Remark 2.19. Definition 2.18 is quite restrictive. In fact, a hard theorem in Topological Group Theory shows that there are **only 11** OCC systems (ignoring translations, reflections, rotations and stretches).

Example 2.20. Spherical polar coordinates (ρ, θ, ϕ):

$$x(\rho,\theta,\phi) = \rho \sin\theta \cos\phi, \qquad y(\rho,\theta,\phi) = \rho \sin\theta \sin\phi, \qquad z(\rho,\theta,\phi) = \rho\cos\theta \ .$$

(a)

Note.

$$arg(x,y) \;:=\; \begin{cases} \tan^{-1}(y/x) & \text{for } \; x>0,\ y\geq 0 \\ \pi/2 & \text{for } \; x=0,\ y>0 \\ \tan^{-1}(y/x)+\pi & \text{for } \; x<0 \\ 3\pi/2 & \text{for } \; x=0,\ y<0 \\ \tan^{-1}(y/x)+2\pi & \text{for } \; x>0,\ y<0 \end{cases}$$

(b)

$$\begin{aligned} \boldsymbol{r}_\rho &= \sin\theta\cos\phi\ \boldsymbol{i} + \sin\theta\sin\phi\ \boldsymbol{j} + \cos\theta\ \boldsymbol{k} \\ \boldsymbol{r}_\theta &= \rho\cos\theta\cos\phi\ \boldsymbol{i} + \rho\cos\theta\sin\phi\ \boldsymbol{j} - \rho\sin\theta\ \boldsymbol{k} \\ \boldsymbol{r}_\phi &= -\rho\sin\theta\sin\phi\ \boldsymbol{i} + \rho\sin\theta\cos\phi\ \boldsymbol{j} \end{aligned}$$

DIY

$$\boldsymbol{r}_\rho \cdot \boldsymbol{r}_\theta \;=$$

$$\boldsymbol{r}_\rho \cdot \boldsymbol{r}_\phi \;=$$

$$\boldsymbol{r}_\theta \cdot \boldsymbol{r}_\phi \;=$$

It is useful to introduce unit vectors

$$\boldsymbol{e}_u \;:=\; \frac{\boldsymbol{r}_u}{|\boldsymbol{r}_u|}, \qquad \boldsymbol{e}_v \;:=\; \frac{\boldsymbol{r}_v}{|\boldsymbol{r}_v|}, \qquad \boldsymbol{e}_w \;:=\; \frac{\boldsymbol{r}_w}{|\boldsymbol{r}_w|}. \tag{2.20}$$

Let $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ be a vector field on Ω. Since $\boldsymbol{e}_u, \boldsymbol{e}_v, \boldsymbol{e}_w$ are orthonormal, we can use them as a **basis** for representing $\boldsymbol{F}$, i.e.

$$\boldsymbol{F} \;=\; F_u\, \boldsymbol{e}_u + F_v\, \boldsymbol{e}_v + F_w\, \boldsymbol{e}_w \tag{2.21}$$

where F_u, F_v, F_w are the **components of $\boldsymbol{F}$** along the coordinate lines $\mathcal{C}_u$, $\mathcal{C}_v$ and $\mathcal{C}_w$.

Addition to previous note. This is why we do not use the notation $\boldsymbol{F} = (F_1, F_2, F_3)^T$ here. It does not carry any information on the coordinate system we work in.

How do we find F_u, F_v and F_w? Note that since $\boldsymbol{e}_u$, $\boldsymbol{e}_v$ and $\boldsymbol{e}_w$ are orthonormal,

$$\boldsymbol{F} \cdot \boldsymbol{e}_u \;=\; F_u\, \boldsymbol{e}_u \cdot \boldsymbol{e}_u + F_v\, \boldsymbol{e}_v \cdot \boldsymbol{e}_u + F_w\, \boldsymbol{e}_w \cdot \boldsymbol{e}_u \;=\; F_u. \tag{2.22}$$

Similarly, $\boldsymbol{F} \cdot \boldsymbol{e}_v = F_v$ and $\boldsymbol{F} \cdot \boldsymbol{e}_w = F_w$.

Example 2.21. Express the vector field $\boldsymbol{F} = z\,\boldsymbol{i}$ in spherical polar coordinate form.

$$\left.\begin{aligned}
|\boldsymbol{r}_\rho| &= \sqrt{\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta} &&= 1\\
|\boldsymbol{r}_\theta| &= \sqrt{\rho^2\{\cos^2\theta(\cos^2\phi + \sin^2\phi) + \sin^2\theta\}} &&= \rho\\
|\boldsymbol{r}_\phi| &= \sqrt{\rho^2\sin^2\theta(\sin^2\phi + \cos^2\phi)} &&= \rho\sin\theta
\end{aligned}\right\} \Longrightarrow$$

$$\Longrightarrow \left\{\begin{aligned}
\boldsymbol{e}_\rho &= \\
\boldsymbol{e}_\theta &= \\
\boldsymbol{e}_\phi &=
\end{aligned}\right.$$

Thus

$$\begin{aligned}
F_\rho &= \boldsymbol{F} \cdot \boldsymbol{e}_\rho = \\
F_\theta &= \boldsymbol{F} \cdot \boldsymbol{e}_\theta = \\
F_\phi &= \boldsymbol{F} \cdot \boldsymbol{e}_\phi =
\end{aligned}$$

and

$$\boldsymbol{F} =$$

2.3.2 Grad, div, curl, ∇^2 in Orthogonal Curvilinear Coordinates

Proposition 2.22. *(a)*

$$\operatorname{grad} f = \frac{1}{|\boldsymbol{r}_u|}\frac{\partial f}{\partial u}\boldsymbol{e}_u + \frac{1}{|\boldsymbol{r}_v|}\frac{\partial f}{\partial v}\boldsymbol{e}_v + \frac{1}{|\boldsymbol{r}_w|}\frac{\partial f}{\partial w}\boldsymbol{e}_w \tag{2.23}$$

(b)

$$\operatorname{div} \boldsymbol{F} = \frac{1}{|\boldsymbol{r}_u||\boldsymbol{r}_v||\boldsymbol{r}_w|}\left\{\frac{\partial}{\partial u}\Big(|\boldsymbol{r}_v||\boldsymbol{r}_w|F_u\Big) + \frac{\partial}{\partial v}\Big(|\boldsymbol{r}_u||\boldsymbol{r}_w|F_v\Big) + \frac{\partial}{\partial w}\Big(|\boldsymbol{r}_u||\boldsymbol{r}_v|F_w\Big)\right\} \tag{2.24}$$

(c)

$$\operatorname{curl} \boldsymbol{F} = \frac{1}{|\boldsymbol{r}_u||\boldsymbol{r}_v||\boldsymbol{r}_w|}\begin{vmatrix} |\boldsymbol{r}_u|\boldsymbol{e}_u & |\boldsymbol{r}_v|\boldsymbol{e}_v & |\boldsymbol{r}_w|\boldsymbol{e}_w \\ \partial/\partial u & \partial/\partial v & \partial/\partial w \\ |\boldsymbol{r}_u|F_u & |\boldsymbol{r}_v|F_v & |\boldsymbol{r}_w|F_w \end{vmatrix} \tag{2.25}$$

(d)

$$\nabla^2 f = \frac{1}{|\boldsymbol{r}_u||\boldsymbol{r}_v||\boldsymbol{r}_w|}\left\{\frac{\partial}{\partial u}\left(\frac{|\boldsymbol{r}_v||\boldsymbol{r}_w|}{|\boldsymbol{r}_u|}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{|\boldsymbol{r}_u||\boldsymbol{r}_w|}{|\boldsymbol{r}_v|}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{|\boldsymbol{r}_u||\boldsymbol{r}_v|}{|\boldsymbol{r}_w|}\frac{\partial f}{\partial w}\right)\right\} \tag{2.26}$$

Proof. (a)

(b) [Problem Sheet 6, Question 1].

(c) See [Bourne, pp. 131–132].

(d)

□

Example 2.23. For a scalar field $f : \Omega \to \mathbb{R}$ find ∇f and $\nabla^2 f$ in spherical polar coordinates.

Recall Example 2.21: $|\boldsymbol{r}_\rho| = 1,\ |\boldsymbol{r}_\theta| = \rho,\ |\boldsymbol{r}_\phi| = \rho\sin\theta$.
Therefore

DIY **Exercise.** Use spherical polar coordinates to show that $f(\boldsymbol{x}) = |\boldsymbol{x}|^{-1}$ satisfies the Laplace Equation $\nabla^2 f = 0$.

Example 2.24. Let $\boldsymbol{F} := \rho^2\cos\theta\ \boldsymbol{e}_\rho + \dfrac{1}{\rho}\ \boldsymbol{e}_\theta + \dfrac{1}{\rho\sin\theta}\ \boldsymbol{e}_\phi$ in spherical polar coordinates. Find curl$\boldsymbol{F}$.

$$\begin{aligned}
\mathrm{curl}\boldsymbol{F} &= \frac{1}{\rho^2\sin\theta}\begin{vmatrix} \boldsymbol{e}_\rho & \rho\,\boldsymbol{e}_\theta & \rho\sin\theta\,\boldsymbol{e}_\phi \\ \partial/\partial\rho & \partial/\partial\theta & \partial/\partial\phi \\ F_\rho & \rho F_\theta & \rho\sin\theta F_\phi \end{vmatrix} \\
&= \frac{1}{\rho^2\sin\theta}\left\{\left(\frac{\partial(1)}{\partial\theta} - \frac{\partial(1)}{\partial\phi}\right)\boldsymbol{e}_\rho + \left(\frac{\partial(1)}{\partial\rho} - \frac{\partial(\rho\sin\theta)}{\partial\phi}\right)\rho\,\boldsymbol{e}_\theta + \left(\frac{\partial(1)}{\partial\rho} - \frac{\partial(\rho^2\cos\theta)}{\partial\theta}\right)\rho\sin\theta\,\boldsymbol{e}_\phi\right\} \\
&= \rho\sin\theta\,\boldsymbol{e}_\phi
\end{aligned}$$