Chapter 3

Integral Theorems

[Anton, pp. 1124–1130, pp. 1145–1160] & [Bourne, pp. 195–224]

First of all some definitions which we will need in the following:

Definition 3.1. (a) A domain (region) Ω is an open connected subset of $\mathbb{R}^n$.

(b) A domain $\Omega \subset \mathbb{R}^3$ is **bounded**, if there exists an $R > 0$ such that $\Omega \subset \mathcal{B}_R$, where $\mathcal{B}_R$ is the ball with radius R and centre $\mathbf{0}$.

(c) A surface $\mathcal{S} \subset \mathbb{R}^3$ is **open**, if for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \notin S$ there exists a continuous curve from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$ which does not cross $\mathcal{S}$. A surface $\mathcal{S} \subset \mathbb{R}^3$ is **closed**, if it is not open.

(d) A closed surface $\mathcal{S} \subset \mathbb{R}^3$ is **convex**, if every straight line intersects (meets) $\mathcal{S}$ at two points at most. **Examples.**

(e) A closed surface $\mathcal{S} \subset \mathbb{R}^3$ is **semi–convex**, if we can choose a coordinate system $0xyz$ so that every straight line *parallel to the coordinate axes* intersects $\mathcal{S}$ at two points at most. **Examples.**

Note. Recall also (Remark 1.24) that a surface $\mathcal{S}$ is smooth, if its parametrisation is continuously differentiable. $\mathcal{S}$ is piecewise smooth, if $\mathcal{S} = \bigcup_{i=1}^{n} \mathcal{S}_i$ and $\mathcal{S}_i$ smooth.

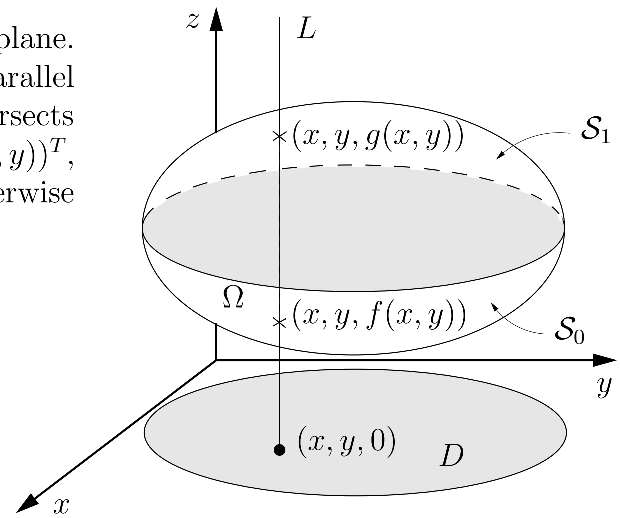
3.1 The Divergence Theorem of Gauss

Theorem 3.2 (Divergence Theorem). *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with piecewise smooth, closed boundary (surface) $\mathcal{S}$. Suppose also that $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ is a continuously differentiable vector field. Then*

$$\iiint_\Omega \nabla \cdot \boldsymbol{F} \, dV \;=\; \iint_{\mathcal{S}} \boldsymbol{F} \cdot \boldsymbol{dS} \;. \tag{3.1}$$

Proof. (Only for $\mathcal{S}$ smooth and semi–convex).

Let D be the projection of Ω onto the (x, y)–plane. Consider the line L through the point $(x, y, 0)$ parallel to the z–axis. Since $\mathcal{S}$ is semi-convex, L intersects $\mathcal{S}$ at two points $(x, y, f(x, y))^T$ and $(x, y, g(x, y))^T$, where $f(x, y) \le g(x, y)$ for all $(x, y) \in D$ (otherwise change the coordinate system).



Hence,

(i) Let us first show that

$$\iint_{\mathcal{S}} F_3 \boldsymbol{k} \cdot \boldsymbol{dS} \;=\; \iint_D \Big\{ F_3(x, y, g(x, y)) - F_3(x, y, f(x, y)) \Big\} \, dx \, dy \;. \tag{3.2}$$

(ii) Now we show that

$$\iiint_\Omega \frac{\partial F_3}{\partial z}\, dV \;=\; \iint_\mathcal{S} F_3 \boldsymbol{k} \cdot \boldsymbol{dS} \;. \tag{3.3}$$

(iii) Similarly, by projecting onto the (x, z)-plane and onto the (y, z)-plane we can establish

$$\iiint_\Omega \frac{\partial F_2}{\partial y}\, dV \;=\; \iint_\mathcal{S} F_2 \boldsymbol{j} \cdot \boldsymbol{dS} \;, \tag{3.4}$$

$$\iiint_\Omega \frac{\partial F_1}{\partial x}\, dV \;=\; \iint_\mathcal{S} F_1 \boldsymbol{i} \cdot \boldsymbol{dS} \;, \tag{3.5}$$

and

□

Remark 3.3. This proof can be extended in a straightforward way to domains Ω with piecewise smooth and non-semi-convex boundary $\mathcal{S}$, if $\Omega = \bigcup_{i=1}^n \Omega_i$, where each of the Ω_i has a smooth, semi-convex boundary $\mathcal{S}_i$, e.g. torus.

Example 3.4. Find $\iint_\mathcal{S} \boldsymbol{F} \cdot \boldsymbol{dS}$ where $\mathcal{S}$ is the surface of the unit cube and $\boldsymbol{F} := (x^2, y^2, z^2)^T$.

Corollary 3.5. *Let Ω and $\mathcal{S}$ be as in Theorem 3.2. Suppose $f : \Omega \to \mathbb{R}$ and $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ are continuously differentiable. Then*

$$\iiint_\Omega \nabla f\, dV \;=\; \iint_\mathcal{S} f\, \boldsymbol{dS} \tag{3.6}$$

$$\iiint_\Omega \nabla \wedge \boldsymbol{F}\, dV \;=\; -\iint_\mathcal{S} \boldsymbol{F} \wedge \boldsymbol{dS} \tag{3.7}$$

Proof. Let $\boldsymbol{a} \in \mathbb{R}^3$ be constant.

(i) Apply the Divergence Theorem to $\boldsymbol{G} := f\,\boldsymbol{a}$:

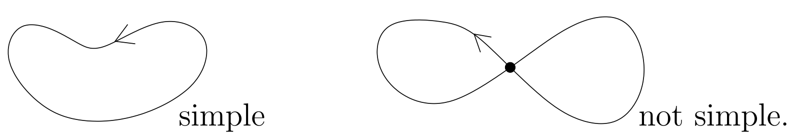
(ii) Apply the Divergence Theorem to $\boldsymbol{G} := \boldsymbol{a} \wedge \boldsymbol{F}$:

□

3.2 Green's Theorem in the Plane

Note. In this section we work in $\mathbb{R}^2$ not in $\mathbb{R}^3$!

Definition 3.6. (a) A closed curve $\mathcal{C} \subset \mathbb{R}^2$, is **simple**, if it does not intersect itself, e.g.



(b) A closed curve $\mathcal{C} \subset \mathbb{R}^2$ is **convex**, if every straight line intersects $\mathcal{C}$ at 2 points at most.

(c) A closed curve $\mathcal{C} \subset \mathbb{R}^2$ is **semi–convex**, if we can choose a coordinate system $0xy$ so that every straight line *parallel to the coordinate axes* intersects $\mathcal{C}$ at 2 points at most.

Theorem 3.7 (Green's Theorem in the Plane). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with simple, piecewise smooth boundary (curve) $\mathcal{C} \subset \mathbb{R}^2$ described in the anticlockwise sense. Suppose that $\boldsymbol{\Phi} : \Omega \to \mathbb{R}^2$ is a continuously differentiable vector field in $\mathbb{R}^2$, i.e. $\boldsymbol{\Phi} = \Phi_1 \boldsymbol{i} + \Phi_2 \boldsymbol{j}$. Then*

$$\iint_\Omega \left(\frac{\partial \Phi_2}{\partial x} - \frac{\partial \Phi_1}{\partial y} \right) dx\, dy \;=\; \oint_{\mathcal{C}} \boldsymbol{\Phi} \cdot d\boldsymbol{r}\ . \tag{3.8}$$

Proof. See **Handout** or [Bourne, pp.210–213]. □

Remark 3.8. Green's Theorem in the plane is sometimes also referred to as **Stokes' Theorem in the plane** (e.g. in [Bourne, pp. 210–213]).

Corollary 3.9. *The area bounded by a simple, closed, piecewise smooth curve $\mathcal{C} \subset \mathbb{R}^2$ is given by*

$$\frac{1}{2} \left| \oint_{\mathcal{C}} (-y\boldsymbol{i} + x\boldsymbol{j}) \cdot d\boldsymbol{r} \right| .$$

Proof. Apply Green's Theorem in the plane with $\Phi_1(x, y) = -y$ and $\Phi_2(x, y) = x$.

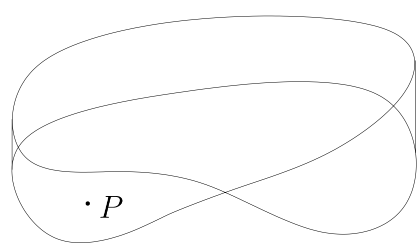
□

3.3 Stokes' Theorem

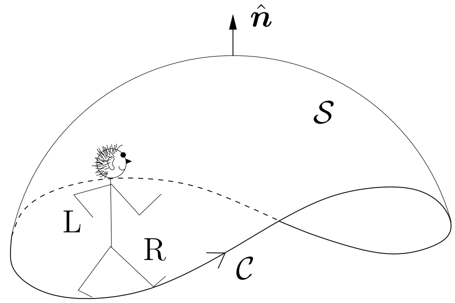
Definition 3.10. (a) A closed curve $\mathcal{C} \subset \mathbb{R}^3$, is **simple**, if it does not intersect itself.

(b) A surface $\mathcal{S} \subset \mathbb{R}^3$ is **orientable**, if a unique normal can be assigned at each point $\boldsymbol{x} \in \mathcal{S}$.

Example. A Möbius strip for example is **not** orientable:



(c) Let $\mathcal{S}$ be an open, orientable surface with simple boundary (curve) $\mathcal{C}$. Let $\hat{\boldsymbol{n}}$ be the unit normal on $\mathcal{S}$. Imagine a person walking along the curve $\mathcal{C}$ (in the positive direction) with its head pointing in the direction of $\hat{\boldsymbol{n}}$.



Then $\mathcal{S}$ and $\mathcal{C}$ are said to be **correspondingly orientated**, if the surface is to the left of the person. [Anton, p. 1154], [Bourne, p. 210].

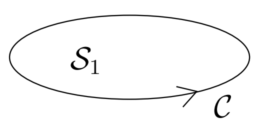
Theorem 3.11 (Stokes' Theorem). *Let $\mathcal{S} \subset \mathbb{R}^3$ be an open, orientable, piecewise smooth surface with correspondingly orientated, simple, piecewise smooth boundary (curve) $\mathcal{C} \subset \mathbb{R}^3$. Suppose that the vector field $\boldsymbol{F}$ is continuously differentiable (in a neighbourhood of $\mathcal{S}$). Then*

$$\iint_{\mathcal{S}} (\nabla \wedge \boldsymbol{F}) \cdot \boldsymbol{dS} \;=\; \oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} \ . \tag{3.9}$$

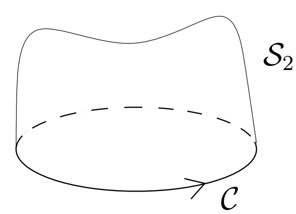
Proof. See **Handout** or [Bourne, pp.213–216]. □

Remark 3.12. (a) Stokes' Theorem implies that the flux of $\nabla \wedge \boldsymbol{F}$ through a surface $\mathcal{S}$ depends only on the boundary $\mathcal{C}$ of $\mathcal{S}$ and is therefore independent of its shape. In other words,

$\iint_{\mathcal{S}} (\nabla \wedge \boldsymbol{F}) \cdot \boldsymbol{dS}$ is the same for



and for



(b) Note that Theorem 3.7 is a special case of Theorem 3.11. To see this, assume that $\mathcal{S}$ in Theorem 3.11 is flat, i.e. $\mathcal{S} \subset \mathbb{R}^2 \times \{0\}$. Then