Revision of first- and second-order linear homogeneous ODEs

In order to solve partial differential equations (PDEs) via the separation of variables method, you need to be familiar with solving ordinary differential equations. Since we restricted ourselves to linear, homogeneous, second-order PDEs it suffices to be able to solve linear, homogeneous, first- and second-order ODEs, i.e.

$$\frac{dy}{dx} + p(x)y = 0, \tag{1}$$

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0, \tag{2}$$

on some open interval I. This was covered in **MA10003**.

The first-order equation (1) is **uniquely** solvable up to a constant factor. If we know the value of y at some point $x = x_0$ then y can be determined uniquely (**initial value problem**). To solve (1) use the **method of integrating factors** (see **MA10003** or *[Anton, Section 9.1]*.

The second-order equation (2) has two linearly independent solutions $y_1(x)$ and $y_2(x)$. Given $y_1(x)$ and $y_2(x)$, a general solution of (2) is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

for some constants c_1 and c_2 (cf. *[Anton, Theorem 9.4.1]*). To fix the values of c_1 and c_2 one needs to prescribe some values on the boundary of I (**boundary value problem**).

When $p(x) = p_0$ and $q(x) = q_0$ are constant for all $x \in I$ then we can find $y_1(x)$ and $y_2(x)$ via the ansatz $y(x) = Ae^{\mu x}$. The two possible choices of μ are the solutions to the **auxiliary equation**

$$\mu^2 + p_0\mu + q_0 = 0 .$$

For more details see your lecture notes to **MA10003** or *[Anton, Section 9.4]*.

When $p(x)$ and $q(x)$ are not constant one can try to reformulate (2) to a new problem with constant coefficients by using an appropriate substitution $x = g(z)$. We will only consider **Euler's differential equation**

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{c}{x^2}y = 0, \tag{3}$$

i.e. $p(x) = 1/x$ and $q(x) = -c/x^2$ and assume $I \subset (x_0, \infty)$ for some $x_0 > 0$. In this case it is useful to use the transformation $x = e^z$. Then

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2} .$$

Hence (3) becomes

$$\frac{d^2y}{dz^2} - cy = 0 . \tag{4}$$

We now need to distinguish the cases $c = 0$ and $c \neq 0$:

Case $c = 0$**.** In this case (4) has general solution $y = A + Bz$ and so after substituting back $z = \ln(x)$ we have

$$y(x) = A\ln x + B .$$

Case $c \neq 0$**.** In this case the auxiliary equation for (4) is $\mu^2 - c = 0$ and so after substituting back $z = \ln(x)$ the general solution of (3) is

$$y(x) = A\,e^{\sqrt{c}\ln x} + B\,e^{-\sqrt{c}\ln x} = A\,x^{\sqrt{c}} + B\,x^{-\sqrt{c}} .$$