Inspirational Example

Flow of air around an aircraft wing may be governed by the PDE:

\[(1 - M^2) \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\]

where

\[M = \text{Mach Number} \begin{cases} \quad M > 1 & \text{supersonic} \\ \quad M < 1 & \text{subsonic} \end{cases}\]

\[u = u(x, y) \quad "potential" \text{ for the flow}\]

**Problem:** Find \( u \) in the region exterior to the wing subject to **boundary conditions**

- on the surface of the wing
- a long way away from the wing
This problem may be solved **approximately** by

- **subdividing** the region of flow into **triangles**,  
- **modelling** the **complicated** solution \( u \) as a **simple** function on each triangle,  
- and solving the arising **system of linear algebraic equations**.

Airfoil mesh with 4253 nodes.

This discretisation leads to a system of linear algebraic equations with 4253 **unknowns**.
Inspirational Example (Aeronautical Engineering)

Now imagine a 3D application (say the new Airbus A380)

- more complex geometry
- using tetrahedra to subdivide region
- huge systems of equations

This is the cutting edge in aeronautical engineering!

Numerical Analysis is necessary for efficient and reliable methods!!
Sedimentary Basin Simulation (Test Case “Congo Offshore”)

Position of the studied area in the Congo offshore

Lithology of the studied 3D-block

* taken from F. Schneider et al., Oil & Gas Science and Technology 55(1), 2000.
Sedimentary Basin Simulation (Mathematical Model)

Conservation of mass:

\[
\begin{align*}
\text{(water)} && \frac{\partial}{\partial t} \left( \rho_W \phi S_W \right) + \text{div} \left( \rho_W \phi S_W \vec{V}_W \right) &= \rho_W q_W \\
\text{(oil)} && \frac{\partial}{\partial t} \left( \rho_O \phi S_O \right) + \text{div} \left( \rho_O \phi S_O \vec{V}_O \right) &= \rho_O q_O
\end{align*}
\]

Conservation of momentum (1D vertical):

\[
\frac{\partial \sigma_z}{\partial z} = \left( \phi (S_W \rho_W + S_O \rho_O) + (1 - \phi) \rho_S \right) g
\]

Rheology (elastoviscoplastic law)†:

\[
\frac{d_s \phi}{d_s t} = -\beta(\phi, \sigma_{\text{eff}}) \frac{d_s \sigma_{\text{eff}}}{d_s t} - \alpha(\phi, \sigma_{\text{eff}}) \sigma_{\text{eff}} \quad \text{with} \quad \sigma_{\text{eff}} := \sigma_z - P_f
\]

together with the constraint

\[
S_W = 1 - S_O
\]

and the generalized Darcy’s law

\[
\begin{align*}
\phi S_W (\vec{V}_W - \vec{V}_S) &= -\eta_W(S_O) K(\phi) \left( \vec{\nabla} P_W - \rho_W g \vec{\nabla} z \right) \quad \text{with} \quad P_W := P_f - S_O P_{\text{cap}}(S_O) \\
\phi S_O (\vec{V}_O - \vec{V}_S) &= -\eta_O(S_O) K(\phi) \left( \vec{\nabla} P_O - \rho_O g \vec{\nabla} z \right) \quad \text{with} \quad P_O := P_f + S_W P_{\text{cap}}(S_O)
\end{align*}
\]

† In the rheology equation \( \frac{d_s \phi}{d_s t} \) represents the total time derivative \( \frac{\partial}{\partial t} \ + \vec{V}_S \cdot \vec{\nabla}(.) \).
Computed oil saturation for the potential reservoir levels at present day.*

* taken from F. Schneider et al., *Oil & Gas Science and Technology* 55(1), 2000.
Groundwater Flow Simulation (Test Case “Sellafield”)

PDE: \[- \nabla \cdot \left( K(x, y) \nabla p \right) = f(x, y) \quad \text{in} \quad \Omega \subset \mathbb{R}^2\]

+ Boundary Conditions

Find \textbf{pressure} \quad p = p(x, y)
and \textbf{fluid velocity} \quad u = K \nabla p(x, y) \quad \text{in} \quad \Omega.

Groundwater Flow Simulation (Numerical Results)