1. Let \( V \) denote the space of functions defined in Lecture 2 and let \( k \in C^1[0,1] \) be uniformly positive. As in lectures, let \( a(v,w) = \int_0^1 kv'w' \), for all \( v, w \in V \). Show that \( a \) is an inner product on \( V \) and write down the induced norm \( \| \cdot \|_a \).

2. Consider the space \( V_h \) from §1.2 in lectures (cf. Problem Sheet 1, Question 2) in the special case of a uniform mesh, i.e. \( x_i = i/(n + 1), i = 0, \ldots, n + 1 \), with \( \phi_i, i = 1, \ldots, n \) denoting the hat functions again. Show that

\[
\int_0^1 \phi_i dx = 1/(n + 1), \quad i = 1, \ldots, n.
\]

Show also that for \( i, j \in \{1, \ldots, n\} \) we have

\[
\int_0^1 \phi'_i \phi'_j dx = \begin{cases} 2(n + 1) & \text{if } i = j, \\ -(n + 1) & \text{if } |i - j| = 1, \\ 0 & \text{otherwise.} \end{cases}
\]

3. Write a MATLAB program to assemble, for any given value of \( n \), the \( n \times n \) tridiagonal matrix:

\[
A = (n + 1) \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 \end{bmatrix}.
\]

Your program should also create the column vector \( f \in \mathbb{R}^n \) with entries \( f_i = 1/(n + 1), i = 1, \ldots, n \). Then using the \( \backslash \) (backslash) command solve the linear system

\[
AU = f
\]

for the solution vector \( U \).

[Hint: Type help diag, help ones and help slash to find out about relevant commands.]

4. Consider the piecewise linear finite element method for the problem

\[-u''(x) = 1, \quad \text{for } x \in (0,1) \quad \text{subject to } u(0) = 0 = u(1).\]

(This is the problem (D) of section (1.1) with \( k(x) = 1 = f(x) \) for all \( x \).) Convince yourself that the program you have constructed in the previous question actually implements this method. The vector \( U \) which comes out of the program contains the values of the approximate solution at the interior mesh points.

By trying a few different values of \( n \) (e.g. \( n = 15, 63 \)), compare a few entries of \( U \) with the exact solution of the differential equation (which you have to find). What do you observe?