Please hand in solutions to Question 1 on Tuesday 23rd March.

1. Let $V$ be a Hilbert space with inner product $(\cdot, \cdot)_V$ and norm $\| \cdot \|_V$. Let $a : V \times V \to \mathbb{R}$ be a bilinear form that satisfies Assumptions (A1)–(A3) from Section 2.4.2 of the lecture notes. Show that $a(\cdot, \cdot)$ is an inner product on $V$ and that the induced (“energy”) norm $\| v \|_a := \sqrt{a(v, v)}$ is an equivalent norm to $\| \cdot \|_V$ in $V$.

2. Let $\tau$ be a triangle with nodes $x^1, x^2, x^3$ and area $\mu(\tau)$. Let $m^1, m^2, m^3$ be the mid-points of the edges of $\tau$. Show that the quadrature rule

$$\int_\tau g \, dx = \frac{\mu(\tau)}{3} \sum_{p=1}^{3} g(m^p).$$

is exact for all quadratic functions on $\tau$.

[Hint: Proceed as in the proof to Lemma 8 and use Question 2(b) on Problem Sheet 4.]