Abstracts

Robust Numerical Upscaling at High Contrast

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(joint work with Daniel Peterseim)

Diffusion processes in heterogeneous porous media are notoriously difficult to approximate accurately if the permeability $\alpha(x)$ of the medium varies over many orders of magnitude and on multiple scales, particularly if the medium undergoes deformations. Modern areas of interest such as hydraulic fracturing, enhanced oil recovery or uncertainty quantification help to compound this problem. Classical homogenisation only works when there is some scale separation, periodicity or ergodicity which rarely is the case in applications. Accurate predictions of flow through such media (with standard discretisation techniques) require the resolution of all major small-scale features. Alternatively, sub grid-scale features need to be taken into account in more costly multiscale schemes. Whether the former or the latter approach is chosen depends on how many simulations need to be carried out through the same medium or through a possibly slightly deformed/modified one (e.g. in a time-dependent multi-phase flow simulation or an optimisation loop).

However, both approaches need robust and efficient coarsening strategies. In the former case, they are needed as the coarse component in multilevel preconditioners that are essential for any scalable and efficient solution of the large-scale linear algebra problems that arise when all features are resolved by the grid. In the latter case, they are needed because standard FE methods will not converge unless the mesh size h is larger than the frequency ε at which the coefficient oscillates. The methodologies that have been developed and analysed are similar, but the difficulties are not entirely the same. Nevertheless, the two research areas have seen a fruitful interaction in recent years with the emergence of new multilevel preconditioners with multiscale coarse spaces lifted from the upscaling literature (e.g. [2]) and novel numerical upscaling techniques based on coarsening strategies from multilevel preconditioners (e.g. [5, 1]). There have even been two very successful Oberwolfach mini-workshops (#0910a and #1307a) on this interaction.

In this talk we consider a promising new numerical upscaling technique, the localisable orthogonal decomposition (LOD) method [4], applied to

$$a(u,v) := \int_\Omega \alpha \nabla u \cdot \nabla v \mathrm{d}x = \int_\Omega g v \mathrm{d}x =: G(v), \quad \text{for all } \ v \in H_0^1(\Omega), \tag{1}$$

with arbitrary heterogeneous coefficient $\alpha_{\min} \le \alpha(x) \le \alpha_{\max}$, without any periodicity or scale separation assumptions. It is a variational multiscale method [3] that uses a selectable quasi-interpolation operator to decompose the solution into a low-dimensional coarse space and a high-dimensional remainder space. The coarse space is spanned by computable basis functions with local support. The localisation is rigorously justified in [4] due to the exponential decay of the "correctors" w.r.t. the standard hat functions. This avoids any artifical localisation boundary conditions, typical for other multiscale methods. For moderate contrast and arbitrary oscillatory coefficients this methodology yields approximations that converge to the true solution at the optimal rate (with respect to the coarse mesh size) without any pre-asymptotic effects.

The promising numerical results in [4] for high-contrast model coefficients are not reflected by the theoretical results for localized bases in that reference, because the physical contrast $\alpha_{\max}/\alpha_{\min}$ enters the error analysis via norm equivalences between energy norm and H^1-seminorm. These equivalences are heavily used to connect variational techniques such as Galerkin orthogonality with approximation properties of standard (coefficient-independent) quasi-interpolation operators. In an upcoming paper [6], we circumvent the critical norm equivalences by using coefficient-dependent quasi-interpolation operators, similar to those in [8], which enjoy optimal approximation properties in α-weighted Sobolev spaces.

Definition. Let $\mathcal{I}_H : V_h \to V_H$ be a linear, continuous interpolation operator from a fine to a coarse piecewise linear FE space, with $h < H$, such that

(QI1) the restriction of $\mathcal{I}_H$ to V_H is an isomorphism,

(QI2) there exists a generic constant C_2, such that for all $v_h \in V_h$ and all $T \in \mathcal{T}_H$,

$$H^{-1}\|\alpha^{1/2}(v_h - \mathcal{I}_H v_h)\|_{L^2(T)} + \|\alpha^{1/2}\nabla(v_h - \mathcal{I}_H v_h)\|_{L^2(T)} \leq C_2\|\alpha^{1/2}\nabla v_h\|_{L^2(\omega_T)}$$

with $\omega_T := \operatorname{int}\left(\bigcup\{K \in \mathcal{T}_H |\, \overline{K} \cap \overline{T} \neq \emptyset\}\right)$.

(QI3) there exists a generic constant C_3, such that for all $v_H \in V_H$ there exists $v_h \in V_h$ with the properties

$$\mathcal{I}_H v_h = v_H, \quad \operatorname{supp} v_h \subset \operatorname{supp} v_H \quad \text{and} \quad \|\alpha^{1/2}\nabla v_h\|_{L^2(\Omega)} \leq \; C_3\|\alpha^{1/2}\nabla v_H\|_{L^2(\Omega)}.$$

This operator gives rise to the L^2-orthogonal decomposition $V_h = V_H \oplus V^{\mathrm{fs}}$ where $V^{\mathrm{fs}} := \ker \mathcal{I}_H$. The key idea to a better approximation is to a-orthogonalise this decomposition. Consider the a-orthogonal projection $\mathcal{P}^{\mathrm{cs}} : V_h \to V^{\mathrm{fs}}$ that maps $v \in V_h$ to the unique solution of

$$a(\mathcal{P}^{\mathrm{cs}}v, w) = a(v, w), \quad \text{for all } w \in V^{\mathrm{fs}}, \tag{2}$$

and define $V^{\mathrm{cs}} := (1 - \mathcal{P}^{\mathrm{cs}})V_H$. Then $V_h = V^{\mathrm{cs}} \oplus V^{\mathrm{fs}}$ and $a(V^{\mathrm{cs}}, V^{\mathrm{fs}}) = 0$.

In practice, we solve localised approximations of the corrector problems (2) for the basis functions Φ_z of V_H, restricting the calculation to nodal patches $\omega_{z,k} \subset \Omega$, centred at the corresponding coarse grid vertex $z \in \mathcal{N}_H$ and $2k$ layers of coarse grid elements wide (see [4, 6] for details). This leads to an approximate multiscale coarse space $V_k^{\mathrm{cs}} \approx V^{\mathrm{cs}}$ with $\dim(V_k^{\mathrm{cs}}) = \dim(V_H)$ and to the upscaled equation

$$a(u_k^{\mathrm{cs}}, v) = G(v), \quad \text{for all } \; v \in V_k^{\mathrm{cs}}. \tag{3}$$

Main Theorem. *If (QI1)–(QI3) are satisfied with constants $C_2 = \mathcal{O}(1) = C_3$ independent of α, and provided $k \gtrsim \log\left(\frac{\alpha_{\max}}{H\,\alpha_{\min}}\right)$ and h is sufficiently small, then*

$$\|\alpha^{1/2}\nabla(u - u_k^{cs})\|_{L^2(\Omega)} \;\lesssim\; \alpha_{\min}^{-1/2}\|g\|_{L^2(\Omega)}\, H. \tag{4}$$

This result does not require any periodicity or scale separation assumptions on the coefficients, and the hidden constants are independent of the contrast $\alpha_{\max}/\alpha_{\min}$.

An example of a quasi-interpolation operator that satisfies assumptions (QI1)–(QI3) with constants $C_2 = \mathcal{O}(1) = C_3$ independent of $\alpha_{\max}/\alpha_{\min}$ for a special class of coefficients that are quasi-monotone on the coarse mesh scale is the quasi-interpolation operator analysed in [8] (or rather a slightly modified version of it. See [8] for a more precise discussion of the type of coefficients that is covered. The constants C_2 and C_3 in our analysis do unfortunately depend on H/ε, but this dependence is not evident in the numerical experiments (see [6]).

In our future work we expect to extend this work to wider classes of coefficients, using different initial coarse spaces, instead of the piecewise linear space V_H, such as those proposed in [1, 7] that enjoy optimal approximation properties in α-weighted Sobolev spaces for arbitrary positive coefficients. The associated quasi-interpolation operators satisfy an assumption similar to (QI2), but it remains to be seen if the whole theory can readily be extended.

References

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