A GENERIC MULTIPLICATION IN QUANTISED SCHUR ALGEBRAS

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ABSTRACT. We define a generic multiplication in quantised Schur algebras and thus obtain a new algebra structure in the Schur algebras. We prove that via a modified version of the map from quantum groups to quantised Schur algebras, defined in [1], a subalgebra of this new algebra is a quotient of the monoid algebra in Hall algebras studied in [10]. We also prove that the subalgebra of the new algebra gives a geometric realisation of a positive part of 0-Schur algebras, defined in [4]. Consequently, we obtain a multiplicative basis for the positive part of 0-Schur algebras.

INTRODUCTION

Schur algebras $S(n, r)$ were invented by I. Schur to classify the polynomial representations of the complex general linear group $GL_n(\mathbb{C})$. Quantised Schur algebras $S_q(n, r)$ are quantum analogues of Schur algebras. Both quantised Schur algebras $S_q(n, r)$ and classical Schur algebras $S(n, r)$ have applications to the representation theory of $GL_n$ over fields of undescribing characteristics.

In [1] A. A. Beilinson, G. Lusztig and R. MacPherson gave a geometric construction of quantised enveloping algebras of type $A$. Among other important results they defined surjective algebra homomorphisms $\theta$, from the integral form of the quantised enveloping algebras to certain finite dimensional associative algebras. They first defined a multiplication of pairs of $n$-step partial flags in a vector space $k^r$ over a finite field $k$ and thus obtained a finite dimensional associative algebra. They also studied how the structure constants behave when $r$ increases by a multiple of $n$. Then, by taking a certain limit they obtained the quantised enveloping algebras of type $A$. J. Du remarked in [5] that the finite dimensional associative algebras studied in [1] are canonically isomorphic to the quantised Schur algebras studied by R. Dipper and G. James in [2].

The aim of this paper is to study a generic version of the multiplication of pairs of partial flags defined in [1]. By this generic multiplication we get an algebra structure in the quantised Schur algebras. We prove that a certain subalgebra of this new algebra is a quotient of the monoid algebra in Hall algebras studied by M. Reineke in [10]. Via a modified version of the surjective algebra homomorphism $\theta$, defined in [1], we prove that the subalgebra is isomorphic to a positive part of 0-Schur algebras, studied by S. Donkin in [4]. Thus we achieve a geometric construction of the positive parts of the 0-Schur algebras.

This paper is organized as follows. In Section 1 we recall definitions and results in [1] on $q$-Schur algebras as quotients of quantised enveloping algebras (see also [5, 6]). In Section 2 we recall definitions and results on the multiplication of pairs of partial flags defined in [1]. By this generic multiplication we get an algebra structure in the quantised Schur algebras. We prove that a certain subalgebra of this new algebra is a quotient of the monoid algebra in Hall algebras studied by M. Reineke in [10]. Via a modified version of the surjective algebra homomorphism $\theta$, defined in [1], we prove that the subalgebra is isomorphic to a positive part of 0-Schur algebras, studied by S. Donkin in [4]. Thus we achieve a geometric construction of the positive parts of the 0-Schur algebras.

1. $q$-SCHUR ALGEBRAS AS QUOTIENTS OF QUANTISED ENVELOPING ALGEBRAS

In this section we recall some definitions and results from [1] on $q$-Schur algebras as quotients of quantised enveloping algebras (see also [5, 6]).
1. q-Schur algebras. Denote by $\Theta_r$ the set of $n \times n$ matrices whose entries are non-negative integers and sum to $r$. Let $V$ be an $r$-dimensional vector space over a field $k$. Let $\mathcal{F}$ be the set of all $n$-steps flags in $V$:

$$V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V.$$  

The group $\text{GL}(V)$ acts naturally by change of basis on $\mathcal{F}$. We let $\text{GL}(V)$ act diagonally on $\mathcal{F} \times \mathcal{F}$. Let $(f, f') \in \mathcal{F} \times \mathcal{F}$, we write

$$f = V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V \quad \text{and} \quad f' = V'_1 \subseteq V'_2 \subseteq \cdots \subseteq V'_n = V.$$  

Let $V_0 = V'_0 = 0$ and define

$$a_{ij} = \dim((V_{i-1} + V_i \cap V'_j) - \dim(V_{i-1} + V_i \cap V'_{j-1}).$$  

Then the map $(f, f') \mapsto (a_{ij})_{ij}$ induces a bijection between the set of $\text{GL}(V)$-orbits in $\mathcal{F} \times \mathcal{F}$ and the set $\Theta_r$. We denote by $\mathcal{O}_A$ the $\text{GL}(V)$-orbit in $\mathcal{F} \times \mathcal{F}$ corresponding to the matrix $A \in \Theta_r$.

Now suppose that $k$ is a finite field with $q$ elements. Let $A$, $A'$, $A'' \in \Theta_r$ and let $(f_1, f_2) \in \mathcal{O}_{A''}$. Following Proposition 1.1 in [1], there exists a polynomial $g_{A,A',A''} = c_0 + c_1 q + \cdots + c_m q^m$, given by

$$g_{A,A',A''} = |\{ f \in \mathcal{F} | (f_1, f) \in \mathcal{O}_A, (f, f_2) \in \mathcal{O}_{A'} \}|,$$

where $c_i$ are integers that do not depend on $q$, the cardinality of the field $k$, and $(f_1, f_2) \in \mathcal{O}_{A''}$.

Now recall that the $q$-Schur algebra $S_q(n, r)$ is the free $\mathbb{Z}[q, q^{-1}]$-module with basis $\{ e_A | A \in \Theta_r \}$, and with an associative multiplication given by

$$e_A e_{A'} = \sum_{A'' \in \Theta_r} g_{A,A',A''} e_{A''}.$$  

For a matrix $A \in \Theta_r$, denote by $\text{ro}(A)$ the vector $(\sum_j a_{1j}, \sum_j a_{2j}, \cdots, \sum_j a_{nj})$ and by $\text{co}(A)$ the vector $(\sum_j a_{1j}, \sum_j a_{2j}, \cdots, \sum_j a_{nj})$. By the definition of the multiplication it is easy to see that

$$e_A e_{A'} = 0 \text{ if } \text{co}(A) \neq \text{ro}(A').$$  

Denote by $E_{ij}$ the elementary $n \times n$ matrix with 1 at the entry $(i, j)$ and 0 elsewhere. We recall a lemma, which we will use later, on the multiplication defined above.

**Lemma 1.1** ([1]). Assume that $1 \leq h < n$. Let $A = (a_{ij}) \in \Theta_r$. Assume that $B = (b_{ij}) \in \Theta_r$ such that $B - E_{h,h+1}$ is a diagonal matrix and $\text{co}(B) = \text{ro}(A)$. Then

$$e_B e_A = \sum_{p: a_{h+1,p} > 0} v^2 \sum_{j > p} a_{hj} \frac{v^{2(a_{h,p}+1)} - 1}{v^2 - 1} e_{A + E_{h,p} - E_{h+1,p}}.$$  

1.2. The map $\theta : U_A(\mathfrak{g}_n) \to S_q(n, r)$. Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ and $v^2 = q$. Let $U_A(\mathfrak{g}_n)$ be the integral form of the quantised enveloping algebra of the Lie algebra $\mathfrak{g}_n$. Denote by $S_q(n, r)$ the algebra $\mathcal{A} \otimes S_q(n, r)$. Let $\theta : U_A(\mathfrak{g}_n) \to S_q(n, r)$ be the surjective algebra homomorphism defined by A. A. Beilinson, G. Lusztig and R. MacPherson in [1]. Through the map $\theta$ we can view the Schur algebra $S_q(n, r)$ as a quotient of the quantised enveloping algebra $U_A(\mathfrak{g}_n)$. We are interested in the restriction of $\theta$ to the positive part $U^+$ of $U_A(\mathfrak{g}_n)$.

Unless stated otherwise, we let $Q$ be the linearly oriented quiver of type $A_{n-1}$:

$$Q : 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n - 1.$$  

By a well-known result of C. M. Ringel (see [11, 12]), the algebra $U^+$ is isomorphic to the twisted Ringel-Hall algebra $H_q(Q)$, which is generated by isomorphism classes of simple $kQ$-modules via Hall multiplication.
Denote by \( S_i \) the simple module of the path algebra \( kQ \) associated to vertex \( i \) of \( Q \). By abuse of notation we also denote by \( M \) the isomorphism class of a \( kQ \)-module \( M \). For any \( s \in \mathbb{N} \), denote by \( D_s \) the set of diagonal matrices satisfying that the entries are non-negative integers and that the sum of the entries is \( s \). For a matrix \( A \in \Theta_r \), denote by \([A] = v^{-\dim \mathcal{O}_A + \dim \mathcal{O}_A} \mathcal{O}_A \mathcal{O}_A\), where \( pr_1 \) is the natural projection to the first component of \( \mathcal{F} \times \mathcal{F} \). Now the map \( F \times F \) is a one-to-one correspondence between \( \text{GL}(b) \) and \( \text{Rep}(b) \). For a matrix \( M \in \text{Rep}(kQ) \) associated to vertex \( i \), we mean the module determined by a matrix \( A \in \Theta_r \), where \( \Theta_r \) is the set of diagonal matrices satisfying that the entries are non-negative integers and that the sum of the entries is \( s \). For a matrix \( A \in \Theta_r \), denote by \([A] = v^{-\dim \mathcal{O}_A + \dim \mathcal{O}_A} \mathcal{O}_A \mathcal{O}_A\), where \( pr_1 \) is the natural projection to the first component of \( \mathcal{F} \times \mathcal{F} \). Now the map \( \theta \) can be defined on the twisted Ringel-Hall algebra as follows:

\[
\theta : H_q(Q) \to \mathcal{O}(n,r), \quad S_i \mapsto \sum_{D \in D_{i+1}} [E_{i,i+1} + D].
\]

2. A monoid given by generic extensions

In this section we briefly recall definitions and results on the monoid of generic extensions in [10], and we let \( k \) be an algebraically closed field. Results in this section work for any Dynkin quiver \( Q = (Q_0, Q_1) \), where \( Q_0 = \{1, \ldots, n\} \) is the set of vertices of \( Q \) and \( Q_1 \) is the set of arrows of \( Q \). We denote by \( \text{mod} \mathcal{O}(kQ) \) and \( \text{Rep}(kQ) \), respectively, the category of \( \text{mod} \mathcal{O}(kQ) \)-modules and the category of finite dimensional representations of \( Q \). We don’t distinguish a representation of \( Q \) from the corresponding \( kQ \)-module.

Let \( b \in \mathbb{N}^{n-1} \). Denote by

\[
\text{Rep}(b) = \Pi_{i \in Q_0} \text{Hom}_k(k^{b_i}, k^{b_i})
\]

the representation variety of \( Q \), which is an affine space consisting of representations with dimension vector \( b \). The group \( \text{GL}(b) = \Pi_i \text{GL}(b_i) \) acts on \( \text{Rep}(b) \) by conjugation and there is a one-to-one correspondence between \( \text{GL}(b) \)-orbits in \( \text{Rep}(b) \) and isomorphism classes of representations in \( \text{Rep}(b) \). Denote by \( \mathcal{E}(M, N) \) the subset of \( \text{Rep}(b) \), containing points which are extensions of \( M \) by \( N \).

**Lemma 2.1** ([10]). The set \( \mathcal{E}(M, N) \) is an irreducible subset of \( \text{Rep}(b) \).

Thus there exists a unique open \( \text{GL}(b) \)-orbit in \( \mathcal{E}(M, N) \). We say a point in \( \mathcal{E}(M, N) \) is generic if it is contained in the open orbit. Generic points in \( \mathcal{E}(M, N) \) are also called generic extensions of \( M \) by \( N \).

**Definition 2.2** ([10]). Let \( M \) and \( N \) be two isomorphism classes in \( \text{mod} \mathcal{O}(kQ) \). Define a multiplication

\[
M \ast N = G,
\]

where \( G \) is the isomorphism class of the generic points in \( \mathcal{E}(M, N) \).

Denote by \( H_q(Q) \) the Ringel-Hall algebra over \( \mathbb{Q}[q] \) and by \( H_0(Q) \) the specialisation of \( H_q(Q) \) at \( q = 0 \).

**Theorem 2.3** ([10]). (1) \( \mathcal{M} = \{\{M|M \text{ is an isomorphism class in } \text{mod} kQ\}, \ast\} \) is a monoid.

(2) \( \mathcal{Q} \mathcal{M} \cong H_0(Q) \) as algebras.

3. A monoid given by a generic multiplication in \( q \)-Schur algebras

In this section we define a generic multiplication in the \( q \)-Schur algebra \( S_q(n, r) \). Via this multiplication we obtain a new algebra in \( S_q(n, r) \). We let \( k \) be an algebraically closed field in this section.

Denote by \( \Theta^u_r = \{A \in \Theta_r|A \text{ is an upper triangular matrix}\} \).

Let \( A, A' \in \Theta^u_r \), define

\[
\mathcal{E}(A, A') = \{(f_1, f_2) \in \mathcal{F} \times \mathcal{F}|3 f \text{ such that } (f_1, f) \in \mathcal{O}_A \text{ and } (f, f_2) \in \mathcal{O}_{A'}\}. 
\]

We denote by \( M(ij) \) the indecomposable representation of \( Q \) with dimension vector \( \sum_{i=1}^{j-1} e_i \), where \( e_i \) is the simple root of \( Q \) associated to vertex \( i \), that is, \( e_i \) is the dimension vector of the simple module \( S_i \). By the module determined by a matrix \( A \in \Theta^u_r \) we mean the module
Let $\bigoplus_{i<j} M(ij)^{a_{ij}}$. Note that for any $(f_1, f_2) \in \mathcal{E}(A, A')$, we have $f_2$ is a subflag of $f_1$. Note also that by omitting the last step of a partial flag $f$ in $\mathcal{F}$, we can view $f$ as a projective $kQ$-module and by abuse of notation we still denote the projective module by $f$. Now for $A \in \Theta^n$, suppose that $(f, h) \in \mathcal{O}_A$ and that $M$ is the module determined by $A$. We have a short exact sequence

$$0 \rightarrow h \rightarrow f \rightarrow M \rightarrow 0,$$

that is, $h \subseteq f$ is a projective resolution of $M$.

Let $b \in \mathbb{N}^{n-1}$ and denote by $k^b$ the $Q_0$-graded vector space with $k^b$ its $i$-th homogeneous component, where $i$ is a vertex of $Q$. Denote by $\text{Hom}_{\mathcal{O}_A}(f, k^b)$ the set of graded linear maps between $f$ and $k^b$, where $f$ is a partial flag in $\mathcal{F}$ viewed as a $Q_0$-graded vector space by omitting its last step.

3.1. **Relation between generic points in $\mathcal{E}(A, A')$ and in $\mathcal{E}(M, N)$**. For $A, A' \in \Theta_r$, we write $A \leq A'$ if $\mathcal{O}_{A'}$ is contained in the Zariski closure of $\mathcal{O}_A$. In this case we say that $(f_1, f_2) \in \mathcal{O}_A$ degenerates to $(f'_1, f'_2) \in \mathcal{O}_{A'}$. Lemma 3.7 in [1] implies the existence of generic points in $\mathcal{E}(A, A')$, in the sense that the closure of their orbit contains orbits of all the other points in $\mathcal{E}(A, A')$. That is, there is a unique open orbit in $\mathcal{E}(A, A')$. In this subsection we will show that there is a nice correspondence between generic points $\mathcal{E}(A, A')$ and generic points in subset $\mathcal{E}(M, N)$, where $M$ and $N$ are the modules determined by $A$ and $A'$, respectively.

Let $(f_1, f_2) \in \mathcal{F} \times \mathcal{F}$ with $f_2$ a subflag of $f_1$. Denote by $a_1$ and $a_2$, respectively, the dimension vectors of the projective modules $f_1$ and $f_2$. Let $b = a_1 - a_2$. We define some sets as follows.

$\text{Inj}(f_2, f_1) = \{\sigma \in \text{Hom}_{kQ}(f_2, f_1) | \sigma \text{ is injective}\};$

$S_1 = \{(\sigma, \eta) \in \text{Inj}(f_2, f_1) \times \text{Hom}_{\mathcal{O}_A}(f_1, k^b) | \eta \text{ is surjective and } \eta \sigma = 0\};$

$S'_1 = \{(\sigma, \eta) \in \text{Hom}_{kQ}(f_2, f_1) \times \text{Hom}_{\mathcal{O}_A}(f_1, k^b) | \eta \text{ is surjective, } \ker \eta \text{ is a } kQ\text{-module and } \eta \sigma = 0\};$

$S_2 = \{\eta \in \text{Hom}_{\mathcal{O}_A}(f_1, k^b) | \eta \text{ is surjective and } \ker \eta \text{ is a } kQ\text{-module}\};$

$S'_2 = \{(M, \eta) \in \text{Rep}(b) \times \text{Hom}_{\mathcal{O}_A}(f_1, k^b) | \eta : f_1 \rightarrow M \text{ is a } kQ\text{-homomorphism}\};$

$\text{Inj}_{A, A'}(f_2, f_1) = \{\sigma \in \text{Inj}(f_2, f_1) | \text{cok}(\sigma) \in \mathcal{E}(M, N)\},$

where $M$ and $N$ are the modules determined by $A$ and $A'$, respectively.

For convenience we denote by $\text{Inj}(f_2, f_1)$ by $S_3$. We obtain some fibre bundles as follows.

**Lemma 3.1** ([8]). The natural projection $\pi_1 : S'_1 \rightarrow S_2$ is a vector bundle.

**Lemma 3.2** ([8]). The natural projection $\pi_2 : S_1 \rightarrow S_3$ is a principal $\text{GL}(b)$-bundle.

**Lemma 3.3** ([8]). The natural projection $\pi_3 : S'_2 \rightarrow \text{Rep}(b)$ is a vector bundle.

Note that any $\eta \in S_2$ determines a unique module $M \in \text{Rep}(b)$ and this defines an open embedding of $S_2$ into $S'_2$. So we can view $S_2$ as an open subset of $S'_2$.

**Lemma 3.4.** (1) $\text{Inj}_{A, A'}(f_2, f_1) = \pi_2(S_1 \cap \pi^{-1}_1(S_3 \cap \pi^{-1}_3(\mathcal{E}(M, N))))$.

(2) $\text{Inj}_{A, A'}(f_2, f_1)$ is irreducible.

**Proof.** Following the definitions of $\pi_1$ and $\pi_3$, $\text{cok} \sigma \in \mathcal{E}(M, N)$ for any $(\sigma, \eta) \in S_1 \cap \pi^{-1}_1(S_3 \cap \pi^{-1}_3(\mathcal{E}(M, N)))$. Therefore $\sigma = \pi_2((\sigma, \eta)) \in \text{Inj}_{A, A'}(f_2, f_1)$. On the other hand, suppose that $\sigma \in \text{Inj}_{A, A'}(f_2, f_1)$. Then $\sigma \in \pi_2(S_1 \cap \pi^{-1}_1(S_3 \cap \pi^{-1}_3(X)))$, where $X$ is the module determined by $\eta$ for a preimage $(\sigma, \eta) \in \pi^{-1}_2(\sigma)$. This proves (1). Now (2) follows from (1) and Lemmas 3.1-3.3. □
Let
\[ \mathcal{E}'(A, A') = \{(f_1, f) \in \mathcal{E}(A, A') | f \in \mathcal{F}\}. \]
Define
\[ \pi : \text{Inj}_{A,A'}(f_2, f_1) \to \mathcal{F} \times \mathcal{F}, \sigma \mapsto (f_1, \text{Im}\sigma), \]
where \(\text{Im}\sigma\) can be viewed as a flag in \(\mathcal{F}\) with the last step the natural embedding of \(\text{Im}\sigma(f_1)_{n-1}\) into \(V\).

Lemma 3.5. (1) \(\text{Im}\pi = \mathcal{E}'(A, A')\).

(2) \(\mathcal{E}'(A, A')\) is irreducible.

Proof. Let \(\sigma \in \text{Inj}_{A,A'}(f_2, f_1)\). Then by the following diagram,

\[
\begin{array}{ccc}
  f_2 & \rightarrow & f_2 \rightarrow 0 \\
  \downarrow \sigma & & \downarrow \\
  \ker \xi \eta & \rightarrow & f_1 \xi \eta \rightarrow M, \\
  \downarrow & & \downarrow \\
  N \rightarrow \cok \sigma & \rightarrow & M \\
\end{array}
\]

where each square commutes and all rows and columns are short exact sequences, we know that \((f_1, \text{Im}\sigma) \in \mathcal{E}'(A, A')\). On the other hand by the definition of \(\mathcal{E}'(A, A')\), for any \((f_1, f) \in \mathcal{E}'(A, A')\), the natural embedding \(f_2 \cong f \subseteq f_1\) is in \(\text{Inj}_{A,A'}(f_2, f_1)\). Now the irreducibility of \(\mathcal{E}'(A, A')\) follows from that of \(\text{Inj}_{A,A'}(f_2, f_1)\). \(\square\)

As a consequence of Lemma 3.5 we can see the existence of a unique dense open orbits in \(\mathcal{E}(A, A')\). Indeed, by Lemma 3.5 and the surjective map \(\mathcal{E}'(A, A') \times \text{GL}(r) \to \mathcal{E}(A, A')\), the set \(\mathcal{E}(A, A')\) is irreducible. Since there are only finitely many \(\text{GL}(r)\)-orbits in \(\mathcal{E}(A, A')\), there exists a unique dense open orbits in \(\mathcal{E}(A, A')\).

Definition 3.6. Let \(\mathcal{O}_{A'}\) be the dense open orbit in \(\mathcal{E}(A, A')\). We say that an injection \(\sigma : f' \to f\) is generic in \(\text{Inj}_{A,A'}(f', f)\) if the pair of flags \((f, \text{Im}\sigma)\) is contained in \(\mathcal{O}_{A'}\).

Proposition 3.7. Let \((\sigma, \eta) \in \mathcal{S}_1, (f_1, f) \in \mathcal{O}_{A'}\) and \((f, f_2) \in \mathcal{O}_{A''}\). Then \(\sigma\) is generic in \(\text{Inj}_{A,A'}(f_2, f_1)\) if and only if the module determined by \(\eta\) is generic in \(\mathcal{E}(M, N)\), where \(M\) and \(N\) are the modules determined by \(A\) and \(A'\), respectively.

Proof. Suppose that \(\mathcal{O}_{A''}\) is the dense open orbit in \(\mathcal{E}(A, A')\) and that \(\mathcal{O}_X\) is the dense open orbit in \(\mathcal{E}(M, N)\). By Lemmas 3.1-3.3, \(\pi_2(S_1 \cap \pi_1^{-1}(S_2 \cap \pi_3^{-1}(\mathcal{O}_X)))\) is open in \(\text{Inj}_{A,A'}(f_2, f_1)\). By Lemma 3.5, \(\pi^{-1}(\mathcal{O}_{A''} \cap \mathcal{E}(A, A'))\) is open in \(\text{Inj}_{A,A'}(f_2, f_1)\). Since \(\text{Inj}_{A,A'}(f_2, f_1)\) is irreducible, the intersection \(\pi_2(S_1 \cap \pi_1^{-1}(S_2 \cap \pi_3^{-1}(\mathcal{O}_X))) \cap \pi^{-1}(\mathcal{O}_{A''} \cap \mathcal{E}(A, A'))\) is non-empty. Therefore, \(X\) is the module determined by \(A''\). This finishes the proof. \(\square\)

3.2. A generic multiplication on \(S_q(n, r)\). We now define a multiplication, called a generic multiplication, by

\[ e_A \circ e_{A'} = \begin{cases} 
  e_{A''} & \text{if } \mathcal{E}(A, A') \neq \emptyset, \\
  0 & \text{otherwise}, 
\end{cases} \]

where \(\mathcal{O}_{A''}\) is the dense open orbit in \(\mathcal{E}(A, A')\).

Proposition 3.8. Let \(A, A', A'' \in \Theta_r^n\). Then \((e_A \circ e_{A'}) \circ e_{A''} = e_A \circ (e_{A'} \circ e_{A''})\)

Proof. By the definition of the multiplication \(\circ\), we see that \((e_A \circ e_{A'}) \circ e_{A''} = 0\) implies that \(e_A \circ (e_{A'} \circ e_{A''}) = 0\) and vice versa. So we may assume that neither of them is zero. Let \(M, N, L\) be the module determined by \(A, A', A''\), respectively. By Lemma 3.1 in [10], we know that \((M \ast N) \ast L = M \ast (N \ast L)\). Now the proof follows from Proposition 3.7. \(\square\)

We can now state the main result of this section.
Theorem 3.9. $Q(\{e_A | A \in \Theta^u_r\}, +, \circ)$ is an algebra with unit $\sum_{D \in D_r} e_D$.

Proof. We need only to show that $\sum_{D \in D_r} e_D$ is the unit. Let $D$ be a diagonal matrix and let $(f_1, f_2) \in \mathcal{O}_D$. Then $f_1 = f_2$. For any $A \in \Theta^u_r$, by the definition of the generic multiplication,

$$e_A \circ \sum_{D \in D_r} e_D = e_A \circ e_C,$$

where $C$ is the diagonal matrix diag$(\sum_j a_{jj}, \cdots, \sum_j a_{nj})$. Since $e_A e_C = \sum_B g_{A,C,B} e_B$, where $(f_1, f) \in \mathcal{O}_B$ and $g_{A,C,B} = |\{|f|((f_1, f) \in \mathcal{O}_A, (f, f) \in \mathcal{O}_C)|\}$, we see that $e_A e_C = e_A$. Therefore $e_A \circ e_C = e_A$. Similarly $(\sum_{D \in D_r} e_D) \circ e_A = e_A$. Therefore $\sum_{D \in D_r} e_D$ is the unit. $\square$

We denote the algebra $Q(\{e_A | A \in \Theta^u_r\}, +, \circ)$ in Theorem 3.9 by $S_0^+$.

4. The algebra $S_0^+$ as a quotient and 0-Schur algebras

We have two tasks in this section. We will first prove that a certain subalgebra of $S_0^+$ is a quotient of the monoid algebra defined in Section 2. We will then prove that this subalgebra gives a geometric realisation of a positive part of 0-Schur algebras.

It is well-known that the specialisation of $S_q(n, r)$ at $q = 1$ gives us the classical Schur algebra $S(n, r)$ of type $\Lambda$. Much about the structure and representation theory of $S(n, r)$ is known, see for example [7]. A natural question is to consider the specialisation of $S_q(n, r)$ at $q = 0$, which is called 0-Schur algebra and denoted by $S_0(n, r)$. 0-Schur algebras have been studied in [4, 9, 13]. In this section the 0-Schur algebras will be studied from a different point of view, that is, via a modified version $\theta : H_q(Q) \rightarrow S_q(n, r)$ of the map $\theta : U_A(g_{il}) \rightarrow S_v(n, r)$.

We call $\theta(H_q(Q))$ the positive part of the $q$-Schur algebra, and denote it by $S^+_q(n, r)$. Denote the specialisation of $S^+_q(n, r)$ at $q = 0$ by $S_0^+(n, r)$. Denote by $S_0^{q+}$ the subalgebra of $S_0^+$, generated by $l_{A,r} = \sum_{D} e_{A+D}$ where $A$ is a strict upper triangle matrix with its entries non-negative integers and the sum is taken over all diagonal matrices in $D_r - \sum_{i,j} a_{ij}$.

4.1. A modified version of $\theta$. For convenience we denote by $E_i$ the element $l_{E_{i,i+1},r}$ in $S_q(n, r)$. We have the following result.

Proposition 4.1. The elements $E_1, \ldots, E_{n-1}$ satisfy the following modified quantum Serre relations:

$$E_i E_j - (q + 1)E_j E_i + q E_j E_i^2 = 0 \text{ for } |i - j| = 1 \text{ and } E_i E_j - E_j E_i = 0 \text{ for } |i - j| > 1.$$

Proof. We only prove the first equation for $j = i + 1$. The remaining part can be done in a similar way. By Lemma 1.1, we have the following.

$$E_i E_{i+1} = l_{E_{i,i+2},r} + l_{E_{i,i+1}+E_{i+1,i+2},r},$$
$$E_{i+1} E_i = l_{E_{i+1,i+2}+E_{i,i+1},r},$$
$$E_i E_i = (q + 1) l_{2E_{i,i+1},r},$$
$$E_i l_{E_{i,i+1}+E_{i+2,i+2},r} = q l_{E_{i,i+1}+E_{i+2,i+2},r},$$
$$E_i l_{E_{i,i+1}+E_{i+1,i+2},r} = l_{E_{i,i+1}+E_{i+1,i+2},r} + (q + 1) l_{2E_{i,i+1}+E_{i+1,i+2},r},$$
$$E_{i+1} E_{i+1} = (q + 1) l_{2E_{i+1,i+1}+E_{i+1,i+2},r}.$$

Therefore,

$$E_i^2 E_{i+1} - (q + 1) E_i E_{i+1} E_i + q E_{i+1} E_i^2 = E_i (E_i E_{i+1} - (q + 1) E_{i+1} E_i) + q E_{i+1} E_i^2 = 0.$$ $\square$
Following Proposition 4.1 and [12], we can now modify the restriction \( \theta_{|H_q(Q)} \) as follows.
\[
\theta : H_q(Q) \to S_q(n,r),
\]
\[
S_i \mapsto E_i
\]
From now on, unless stated otherwise, by \( \theta \) we mean the modified map \( \theta_{|H_q(Q)} \). The following result is a modified version of Proposition 2.3 in [6] and we will give a direct proof. For any two modules \( M, N \), recall that Hall multiplication of \( M \) and \( N \) is given by
\[
MN = \sum_X F_{MN}^X X,
\]
where \( F_{MN}^X = |\{U \subseteq X | U \cong N, X/U \cong N\}| \) and where the sum is taken over all the isomorphism classes of modules.

**Proposition 4.2.** Let \( A \) be a strict upper triangular matrix with entries non-negative integers and let \( M \) be the module determined by \( A \). Then
\[
\theta(M) = \begin{cases} 
 l_{A,r} & \text{if } \sum_{i,j} a_{ij} \leq r, \\
 0 & \text{otherwise}.
\end{cases}
\]

**Proof.** Note that \( B = \{ \Pi_{ij} M(ij)^2x_{ij} | x_{ij} \in \mathbb{Z}_{\geq0}\} \) is a PBW-basis of \( H_q(Q) \), where the product is ordered as follows: \( M(ij) \) is on the left hand side of \( M(st) \) if either \( i = s \) and \( j > t \), or \( i > s \). Let \( M \in B \) be the module determined by \( A \). Suppose that \( M(st) \) is the left most term with \( x_{st} > 0 \). We can write \( M = M(st) \oplus M' \).

First consider the case \( M = M(st) \), that is, \( M \) is indecomposable. We may suppose that \( t - s > 1 \). Then \( M = M(s,t-1)M(t-1,t) = M(t-1,t)M(s,t-1) \). By induction on the length of \( M \), Lemma 1.1 and a dual version of it, we have
\[
\theta(M(s,t-1)M(t-1,t)) = l_{E_{s,t-1},r_1} + l_{E_{s,t-1} + E_{t-1,t},r_2},
\]
\[
\theta(M(t-1,t)M(s,t-1)) = l_{E_{s,t-1} + E_{t-1,t},r_3}.
\]
Therefore
\[
\theta(M) = \theta(M(s,t-1)M(t-1,t)) - \theta(M(t-1,t)M(s,t-1)) = l_{E_{s,t},r}.
\]

Now consider the case that \( M \) is decomposable. We use induction on the number of indecomposable direct summands of \( M \). By the assumption we have \( M = \frac{1}{q^{|E_{st}|}}M(st)M' \).

Therefore
\[
\theta(M) = \frac{q^{|E_{st}|} - 1}{q^{|E_{st}|}} \theta(M(st)) \theta(M')
\]
\[
= \frac{q^{|E_{st}|} - 1}{q^{|E_{st}|}} \sum_{D \in D_{|E_{st}|}} e^{E_{st}+D} e^{E_{st}+D'}
\]
where \( D' \) is the diagonal matrix with non-negative integers as entries such that \( co(E_{st}+D) = ro(A - E_{st} + D') \). Suppose that \( e_B \) appears in the multiplication of \( e^{E_{st}+D} e^{E_{st}+D'} \) and \( (f,h) \in O_B \). Note that in the minimal projective resolution \( Q \to P \) of \( M' \), the projective module \( P_i \) is not a direct summand of \( P \). Therefore by the definition of the multiplication \( e^{E_{st}+D} e^{E_{st}+D'} \), we have \( f/h \cong M \), that is \( B = A \), and the coefficient of \( e_B \) is about the possibilities of choosing a submodule, which is isomorphic to \( P_s \), of \( P_s^{E_{st}} \). Hence the coefficient is \( \frac{q^{|E_{st}|} - 1}{q^{|E_{st}|}} \) and so \( \theta(M) = \sum_D e^{E_{st}+D} e^{E_{st}+D'} \). This finishes the proof. \( \square \)

**Remark 4.3.** Proposition 2.3 in [6] has a minor inaccuracy. Indeed, there is a coefficient missing in front of the image \( \theta(M) \). For example, let \( n = 3 \), \( r = 2 \) and let \( M \) be the module determined by the elementary matrix \( E_{13} \). Then \( \theta(M) = vy_{X,r} \), but not \( y_{X,r} \), as stated in Proposition 2.3 in [6], here \( \theta \) is the original map from the quantised enveloping algebra to the Schur algebra \( S_{v}(3,2) \) and \( y_{X,r} = \sum_{D \in D_1} |X + D| \).
4.2. A homomorphism of algebras $\Gamma : QM \to S_0^+$. For a given module $M$, denote by $|M|_{\text{dir}}$ the number of indecomposable direct summands of $M$. Let $\Gamma : QM \to S_0^+$ be the map given by

$$
\Gamma(M) = \begin{cases} 
\theta(M) & \text{if } |M|_{\text{dir}} \leq r, \\
0 & \text{otherwise}.
\end{cases}
$$

Let $X$ be a module and let $D = \text{diag}(d_1, \cdots, d_n)$ be a diagonal matrix. Write $X = \bigoplus_{i,j} M(ij)^{a_{ij}}$. By $e_{X+D}$ we mean the basis element in $S_0(n, r)$ corresponding to the matrix with its entry at $(i, j)$ given by $x_{ij} + \delta_{ij}d_i$ for $i \leq j$ and 0 elsewhere, where $\delta_{ij}$ are the Kronecker data. Let $\sigma = (\alpha_i)_i : Q \to P$ be an injection of $Q$ into $P$, where $P$ and $Q$ are projective modules. Then $(P, Q)$ gives a pair of flags $(f_1, f_2)$ with $f_2$ a subflag of $f_1$. More precisely, the $i$-th step of $f_1$ is given by $\text{Im}P_{a_{n-2}} \cdots P_{a_i}$ for $i \leq n - 2$ and the $(n - 1)$-th step is given by the vector space associated to vertex $n - 1$ of $P$, where $P_{a_j}$ is the linear map on the arrow $\alpha_j$ from $j$ to $j + 1$ for the module $P$. The $i$-th step of $f_2$ is given by $\text{Im}P_{a_{n-2}} \cdots P_{a_i} \sigma_j$ for $i \leq n - 2$ and the $(n - 1)$-th step is given by $\text{Im}\sigma_{n-1}$. We have the following result.

**Theorem 4.4.** The map $\Gamma$ is a morphism of algebras.

**Proof.** The unit in $\mathcal{M}$ is the zero module. By the definition of $\Gamma$, it is clear that $\Gamma(0) = \sum_{D \in D_r} e_D$, the unit of $(\Theta^a, \circ)$.

We now need only to show

$$
\Gamma(M \ast N) = \Gamma(M) \circ \Gamma(N). \tag{1}
$$

Let $X$ be a generic point $\mathcal{E}(M, N)$. Denote the number of indecomposable direct summands of $X$, $M$, $N$ by $a$, $b$ and $c$, respectively. By the definition of $\Gamma$, we can write

$$
\Gamma(X) = \sum_{D \subseteq D_r-a} e_{X+D},
$$
$$
\Gamma(M) = \sum_{D' \subseteq D_r-b} e_{M+D'},
$$
$$
\Gamma(N) = \sum_{D'' \subseteq D_r-c} e_{N+D''}.
$$

We first consider the case where $a > r$. Clearly, in this case $\Gamma(M \ast N) = 0$ and we claim that $e_{M+D'}e_{N+D''} = 0$ for any $D' \subseteq D_r-b$ and $D'' \subseteq D_r-c$. In fact suppose that $e_{L+D''}$ appears in the multiplication, where $L$ is a module and $D''$ is a diagonal matrix. Then $|L|_{\text{dir}} \leq r$, and $L$ is a degeneration of $X$. Since $Q$ is linearly oriented, $|L|_{\text{dir}} \geq a$. This is a contradiction. Therefore $e_{M+D'}e_{N+D''} = 0$, and so

$$
(\sum_{D' \subseteq D_r-b} e_{M+D'}) \circ (\sum_{D'' \subseteq D_r-c} e_{N+D''}) = 0.
$$

This proves the equation (1) for the case $a > r$.

Now suppose that $a \leq r$. Note that if $(D', D'') \neq (C', C'')$, where $D', C' \subseteq D_r-b$ and $D'', C'' \subseteq D_r-c$, then $e_{M+D'} \circ e_{N+D''} \neq e_{M+C'} \circ e_{N+C''}$. By Proposition 3.7, we know that if $e_{M+D'} \circ e_{N+D''} \neq 0$, then $e_{M+D'} \circ e_{N+D''} = e_{X+D}$ for some $D \subseteq D_r-a$.

On the other hand, we can show that for any $e_{X+D}$ appearing in the image of $\Gamma$, there exist $D' \subseteq D_{r-b}$ and $D'' \subseteq D_{r-c}$ such that $e_{M+D'} \circ e_{N+D''} = e_{X+D}$. Suppose that

$$
\begin{array}{c}
\xymatrix{
0 & Q 
\ar[r]^\sigma & P 
\ar[r]^\tau & X 
\ar[r]^0 & 0
}
\end{array}
$$

is the minimal projective resolution of $X$. Write $D = \text{diag}(d_1, \cdots, d_n)$ and let $Y$ be the projective module $\bigoplus_{i,j} P_i^{a_{ij}}$. Then the pair of flags in $\mathcal{F} \times \mathcal{F}$, determined by $(P \oplus Y, Q \oplus Y)$, is in the orbit $\mathcal{O}_{X+D}$. We have the following diagram where each square commutes and all rows and columns are short exact sequences,
\[
\begin{array}{ccc}
\text{Ker}\lambda & \longrightarrow & Q \oplus Y \longrightarrow 0 \\
\downarrow & & \downarrow \\
K & \longrightarrow & P \oplus Y \xrightarrow{(pr\ 0)} M, \\
\downarrow & & \downarrow \\
N \xrightarrow{i} X \xrightarrow{p} M
\end{array}
\]

where \(I\) is the identity map on \(Y\), \(K = \text{Ker}(pr\ 0)\) and \(\lambda = (\tau(0)|_K)\). Let \(Z' \xrightarrow{I} Z'\) be the maximal contractible piece of the projective resolution

\[
0 \longrightarrow K \longrightarrow P \oplus Y \xrightarrow{(pr\ 0)} M \longrightarrow 0
\]

of \(M\), and let \(Z'' \xrightarrow{I} Z''\) be the maximal contractible piece of the projective resolution

\[
0 \longrightarrow \text{Ker}\lambda \longrightarrow K \longrightarrow N \longrightarrow 0
\]

of \(N\). Write

\(Z' = \oplus_i P'd'\) and \(Z'' = \oplus_i P''i\),

and let

\(D' = \text{diag}(d'_1, \cdots, d'_n)\) and \(D'' = \text{diag}(d''_1, \cdots, d''_n)\).

Then \(e_{M+D'} \circ e_{N+D''} = e_{X+D}\). Therefore,

\[
\sum_{D' \in D_{r-\beta}} e_{M+D'} \circ \sum_{D'' \in D_{r-\gamma}} e_{N+D''} = \sum_{D \in D_{r-\alpha}} e_{X+D}.
\]

This proves the equations (1), and so the proof is done. \(\Box\)

The following result is a direct consequence of Theorem 4.4.

**Corollary 4.5.** \(\text{Ker}\Gamma = \text{Q-Span}\{M||M|_{\text{dir}} > r\}\).

**4.3. A geometric realisation of 0-Schur algebras.**

**Theorem 4.6.** \(S^+_0(n, r) \cong \text{Q}S^+_0\) as algebras.

**Proof.** Denote by \(\theta_0\) the specialisation of \(\theta\) to 0, that is, \(\theta_0 : H_0(Q) \to S_0(n, r)\). We have \(\text{Ker}\theta_0 = \text{Q-Span}\{M||M|_{\text{dir}} > r\} = \text{Ker}\Gamma\), where \(\Gamma\) as in Theorem 4.4. Now the proof follows from the following commutative diagram.

\[
\begin{array}{ccc}
\text{Ker}\theta_0 & \longrightarrow & H_0(Q) \xrightarrow{\theta_0} S_0^+(n, r) \\
\downarrow \cong & & \downarrow \cong \\
\text{Ker}\Gamma & \longrightarrow & \mathcal{M} \xrightarrow{\Gamma} S^+_0
\end{array}
\]

\(\Box\)

As a direct consequence of Theorem 4.6, we obtain a multiplicative basis of the positive part of 0-Schur algebras, in the sense that the multiplication of any two basis elements is either a basis element or zero.

**Corollary 4.7.** The elements in \(\{l_A, r| A \text{ is an strictly upper triangular matrix in } \bigcup_{s \leq r} \Theta^n_s\}\) form a multiplicative basis of \(S_0^+(n, r)\).
Under the map $\Gamma$, this multiplicative basis $\{l_{A,r}\}$ for any $A \in \bigcup_{s \leq r} \Theta^u_{s}$ is the image of the multiplicative basis for $H_0(Q)$ studied in [10]. By Theorem 7.2 in [10], the multiplicative basis for $H_0(Q)$ is the specialisation of Lusztig’s canonical basis for a two-parameter quantization of the universal enveloping algebra of $\mathrm{gl}_n$ given in [14]. Thus we can consider the basis $\{l_{A,r}\}$ $A$ is a strictly upper triangular matrix in $\bigcup_{s \leq r} \Theta^u_{s}$ as a subset of a specialization of the canonical basis.

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References


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