Johannes Zimmer Second Semester 2011–2012
Problem set 2

MA30170/MA50170 Numerical Solution of PDEs I

H6. Let V denote the space of functions defined in Lecture 1 and let $k \in C^1[0,1]$ be uniformly positive. As in the lectures, let $a(v,w) = \int_0^1 kv'(x)w'(x)\,\mathrm{d}x$, for all $v, w \in V$. Show that a is an inner product on V and write down the induced norm $\|\cdot\|_a$.

H7. Show that under the assumptions of Section 1.4,

$$\left\{\int_0^1 \left|(u(x) - u_h(x))'\right|^2 \,\mathrm{d}x\right\}^{1/2} \le Ch\,\left|u''\right|_\infty , \quad \text{with} \quad C = \left\{\frac{\max_{x\in[0,1]} k(x)}{\min_{x\in[0,1]} k(x)}\right\}^{1/2} .$$

Look up the Fundamental Theorem of Calculus (**MA20218**) (be prepared to answer questions on it in the Problem Class!). Using the Fundamental Theorem and the bound you proved above, deduce that $\|u - u_h\|_\infty = O(h)$ as $h \to 0$, provided $u \in C^2[0,1]$.

Hint: An integral of a function v can be bounded by the square root of an integral of v^2 by using the Cauchy-Schwarz inequality in a clever way. You will need this for $v = (u - u_h)'$.

H8. Here you need to remember Taylor's Theorem from **MA10207**. The estimate in the previous question may be pessimistic. Show that, provided $u \in C^2[0,1]$, then $|u - \Pi_h u|_\infty \le Ch^2\,|u''|_\infty$ for some constant C independent of h.

Note: This shows that we can reasonably expect that the finite element method may converge with $O\left(h^2\right)$ in the uniform norm. We shall prove this later.

Hint: Consider $p_i(x) = u\left(x_{i-1}\right) + \left(x - x_{i-1}\right) u'\left(x_{i-1}\right)$ and write $u - \pi_h u = (u - p_i) - (\Pi_h u - p_i)$. Use Taylor's theorem to show that

$$\max_{x\in[x_{i-1},x_i]} |u(x) - p_i(x)| \le \frac{1}{2}h^2 \max_{x\in[x_{i-1},x_i]} \left|u''(x)\right| .$$

Then derive the result.

H9. Recall the trapezoidal rule for approximately integrating a function g over an interval $[a,b]$:

$$\int_a^b g(x)\,\mathrm{d}x \approx \frac{(b-a)}{2}\{g(a) + g(b)\}$$

(revise **MA20222**). Show that this approximation is exact in the special case when g is a linear function (i.e., has a straight line graph) on $[a,b]$.

Please hand in solutions to H7. and H8. on Tuesday 21st February.