MA30170/MA50170 Numerical Solution of PDEs I

The first two questions are about extending the program discussed in **H4**. and **H5**. of Sheet 1 to the more general problem

$$-\left(ku'\right)'(x) = f(x)\ ,\ x \in (0,1) \qquad u(0) = 0 = u(1)\ . \tag{1}$$

H10. Consider problem (1) in the special case $k(x) = 1/(2+x)$, $f(x) = 1$, for $x \in [0,1]$. The piecewise linear finite element method leads to a linear system of the form $A\mathbf{U} = \mathbf{f}$. The vector $\mathbf{f}$ was worked out already (see **H5**.). Using the formulae given in class, work out analytic expressions for the non-zero entries of the tridiagonal matrix A.

Then extend the code which you wrote for exercise **H4**. to compute the finite element solution for this problem. For this problem the exact solution u can be found analytically. Find it. Then compute a table of values of the error indicator $\epsilon_n := \max_{i=1,\ldots,n} |u(x_i) - u_h(x_i)|$ against n, for $n = 16, 32, 64, \ldots$. What is the approximate rate of convergence of ϵ_n to zero as $n \to \infty$? (*Hint:* To compute approximate rates of convergence, you can conjecture that $\epsilon_n = C\,(1/n)^\alpha$, for some unknown constants C and α. Then $\epsilon_n/\epsilon_{2n} = 2^\alpha$. Approximations to α can be computed and the rate of convergence of ϵ_n is then $\approx O\,(1/n^\alpha)$.)

H11. Now consider (1) in the case of general k and f. When this is solved by the FEM, it is often not possible to compute the entries of A or $\mathbf{f}$ by analytic integration, and numerical integration must be used. In all cases the required entries of A and $\mathbf{f}$ can be expressed as (sums of) integrals over subintervals of the mesh. For example:

$$A_{i,i} \;=\; \int_{x_{i-1}}^{x_i} k\varphi_i'(x)\varphi_i'(x)\,\mathrm{d}x \;+\; \int_{x_i}^{x_{i+1}} k\varphi_i'(x)\varphi_i'(x)\,\mathrm{d}x\ .$$

The integrals on the right-hand side can be approximated by the trapezoidal rule applied to each of the subintervals $[x_{i-1}, x_i]$.

Extend your finite element program so that it solves the general problem (1) and uses the trapezoidal rule over each subinterval to compute the entries of A and $\mathbf{f}$. The result will be formulae involving the values of k and f. The neatest way to do this will be to create function files to evaluate k and f at a given point $x \in [0,1]$ which are stored separately and then called from the main program. This makes it easy to change k and f and to run different examples.

Test your program in the case $k(x) = 1/(2+x)$ and $f(x) = 1$. How do the errors ϵ_n compare with those in exercise **H10**.? What is the rate of convergence?

H12. A two-dimensional example. Consider the (reference) triangle $\hat{\tau}$ with vertices $(0,0)$, $(1,0)$, $(0,1)$. Find weights $\hat{w}_1, \hat{w}_2, \hat{w}_3$ such that the quadrature rule

$$\int_{\hat{\tau}} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \;\approx\; \hat{w}_1 f(0,0) + \hat{w}_2 f(1,0) + \hat{w}_3 f(0,1)$$

is exact for linear functions on $\hat{\tau}$ (that is, functions of $\mathbf{x} = (x_1, x_2)$ which are linear combinations of 1, x_1 and x_2). *Hint:* Derive three linear equations which must be satisfied by the weights and solve them.

Please hand in solutions to H11. on Tuesday 28th February.