MA30170/MA50170 Numerical Solution of PDEs I

H21. Show that if τ is any triangle, then, for any continuous function $v\colon \tau \to \mathbb{R}$, there is one and only one linear function which interpolates v at the three nodes of this triangle. *Hint:* write any interpolant in the form $\sum_{p=1}^{3} \nu^p \varphi^p$ and consider what happens if there are two such interpolants.

H22. Let τ be a triangle with nodes $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3$ and area $\mu(\tau)$. Let $\mathbf{m}^1, \mathbf{m}^2, \mathbf{m}^3$ be the midpoints of the edges of τ. Show that the quadrature rule

$$\int_\tau g(\mathbf{x})\,\mathrm{d}\mathbf{x} \;=\; \frac{\mu(\tau)}{3}\sum_{p=1}^{3} g\left(\mathbf{m}^p\right)$$

is exact for all quadratic functions on τ. *Hint:* Proceed as in the proof of Lemma 11 and use Exercise **H14**. on Sheet 4.

H23. (2002/03 exam, Question 4) Let Ω be a domain on $\mathbb{R}^2$ with a polygonal boundary Γ, and suppose Ω is decomposed into a mesh of triangular elements $\mathcal{T}$.

For any $\tau \in \mathcal{T}$, let $\mathbf{x}^p,\ p = 1, 2, 3,$ denote its nodes (ordered anticlockwise), let h_τ denote its diameter and let $\mu(\tau)$ denote its area. Let $\varphi^p,\ p = 1, 2, 3,$ denote the linear functions on τ, with the property

$$\varphi^p(\mathbf{x}^q) = \begin{cases} 1 & \text{when } p = q\ , \\ 0 & \text{when } p \neq q\ . \end{cases}$$

Let $\mathbf{e}_j$ denote the jth standard basis vector in $\mathbb{R}^2$, $j = 1, 2$.

a) Give a formula for the function $\pi_\tau v$, the linear function which interpolates v at each of the nodes $\mathbf{x}^p$.

b) Give, without proof, formulae for the sums:
   - $\sum_{p=1}^{3} \frac{\partial \varphi^p}{\partial x_j}(\mathbf{x})$ for $\mathbf{x} \in \tau$,
   - $\sum_{p=1}^{3} \frac{\partial \varphi^p}{\partial x_j}(\mathbf{x})\, \mathbf{x}^p$ for $\mathbf{x} \in \tau$.

c) Deduce that, for any $v \in C^2(\tau)$,

$$\left\| \frac{\partial}{\partial x_j}\left(v - \pi_\tau v\right) \right\|_{\infty,\tau} \leq C \frac{h_\tau^3}{\mu(\tau)} \max_{|\alpha|=2} \|D^\alpha v\|_{\infty,\tau}\ ,$$

where C is a constant independent of the mesh and of v.

[You may assume the Taylor expansion: $v(\mathbf{y}) = v(\mathbf{x}) + \nabla v(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) + R(\mathbf{y}, \mathbf{x})$, for $\mathbf{x}, \mathbf{y} \in \tau$, with $|R(\mathbf{y}, \mathbf{x})| \leq C h_\tau^2 \max_{|\alpha|=2} |D^\alpha v|_{\infty,\tau}$. You may also use without proof a suitable bound on $\left\|\partial \varphi^p / \partial x_j\right\|_{\infty,\tau}$.]

d) Give the definition of a *regular* collection of meshes.

Prove that, for a regular collection of meshes, there exists a constant C independent of the meshes such that

$$h_\tau \leq C h_{\tau'}$$

for all pairs of triangles τ, τ' which intersect at a common edge.

Please hand in solutions to H21. and H22. on Tuesday 12th April.