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Homework Assignment #1

AM 125c

Engineering Mathematical Principles

H1. *(7 points).* $\mathrm{Re}(\lambda) < 0$ *does not imply stability for nonautonomous systems*
In AM125b, we have seen that $\mathrm{Re}(\lambda) < 0$ implies stability for the *autonomous* system $\dot{x}(t) = Ax(t)$. The following counterexample due to MARCUS (1960) shows this is not true for the *nonautonomous* system

$$\dot{x}(t) = A(t)x(t), \text{ where } A(t) = \begin{pmatrix} -1 + 3\cos(8t) & 4 - 3\sin(8t) \\ -4 - 3\sin(8t) & -1 - 3\cos(8t) \end{pmatrix}.$$

To see this, show that

a) $\chi(\lambda) = \det(A(t) - \lambda I)$ is stable for all $t \in \mathbb{R}$.

b) A solution is given by

$$x(t) = \exp(2t) \begin{pmatrix} -\cos(4t) \\ \sin(4t) \end{pmatrix}.$$

c) $x = 0$ is an unstable equilibrium point.

H2. *(7 points).* Consider the system

$$\begin{aligned} \dot{x} &= -y + xy - x^3 - \frac{1}{2}xy^2 \\ \dot{y} &= -3y + xy + x^2y - \frac{1}{2}xy^2. \end{aligned}$$

a) What can you conclude about the stability of the equilibrium solution $x = y = 0$ for this system using the Lyapunov-Poincaré Theorem?

b) Use $V(x, y) := \frac{1}{2}(3x^2 - 2xy + y^2)$ as a Lyapunov function to show the origin is stable.

H3. *(10 points).*

a) Prove Lemma (1.4): The Floquet multipliers for

$$\dot{x}(t) = A(t)x(t), \tag{1}$$

$A(t)$ periodic with period T, are independent of the choice of the fundamental matrix $U(t)$: two fundamental matrices $U(t), \widetilde{U}(t)$ of (1) give the same Floquet multipliers.

b) Show that for the Floquet multipliers λ_j of (1) and the characteristic exponents μ_j,

$$\prod_{j=1}^{n} \lambda_j = \exp\left(\int_0^T \mathrm{tr}(A(t))\mathrm{d}t\right)$$

and

$$\sum_{j=1}^{n} \mu_j = \frac{1}{T}\int_0^T \mathrm{tr}(A(t))\mathrm{d}t \qquad \mod \frac{2\pi i}{T}.$$

Remark. This can be used to compute the last Floquet multiplier if $n-1$ Floquet multipliers are known.

H4. *(10 points).* Maple session:

a) Learn about the different numerical solution techniques available in maple (use `?dsolve,numeric`). Use different methods (explicit and implicit, extrapolation,...—at least 3 algorithms) to approximate a stiff equation with suitably chosen initial conditions. Evaluate the different algorithms based on that experience.

b) Construct at least two more tough problems to compare the quality of different algorithms (here, the knowledge from AM125b might be helpful: think carefully to construct real torture tests for algorithms). Explain your choice and evaluate at least three different algorithms.