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Homework Assignment #2

AM 125c
Engineering Mathematical Principles

H5. *(7 points).* Consider the second order differential equation

$$\ddot{u}(t) + 2a(t)\dot{u}(t) + b(t)u(t) = 0, \tag{1}$$

where a and b are real, continuous and periodic functions with period T.

a) Rewrite this as a first order system

$$\dot{y}(t) = A(t)y(t).$$

Assume that u, v are solutions of (1) with $u(0) = 1, \dot{u}(0) = 0$ and $v(0) = 0, \dot{v}(0) = 1$. Express the transition matrix C as a function of u and v. Determine $\det(C)$ as a function of $a(t)$.

b) Compute the Floquet multipliers as a function of $\operatorname{tr}(C)$ and $\det(C)$.

c) What can you say about existence of (real or complex) normal solutions, depending on $\operatorname{tr}(C)$ and $\det(C)$?

H6. *(10 points). Poisson's formula*
Suppose $B(0, r_0)$ is the disk with radius r_0 and center 0 in the plane. We want to show that the solution of

$$\begin{aligned} \Delta u &= 0 \text{ in } B(0, r_0), \\ u &= f \text{ on } \partial B(0, r_0) \end{aligned}$$

is for continuous f given by *Poisson's integral formula*

$$u(r, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} f(t) \frac{r_0^2 - r^2}{r_0^2 - 2r_0 r \cos(t - \varphi) + r^2} \mathrm{d}t$$

$(r < r_0)$.

a) By separation of variables or otherwise, show that the functions

$$u_n(r, \varphi) = r^n \left(A_n \cos(n\varphi) + B_n \sin(n\varphi)\right), \qquad n = 0, 1, 2, \ldots$$

satisfy Laplace's equation.

b) By superimposing these solutions, find a formal solution of the boundary value problem (use that f expressed in polar coordinates is a periodic function which can be represented as a Fourier series)

c) Show this formal solution can be rewritten as

$$u(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\left\{1 + 2\sum_{n=1}^{\infty}\cos\left(n(t-\varphi)\right)\left(\frac{r}{r_0}\right)^n\right\}\mathrm{d}t.$$

d) By summing the series, find Poisson's integral formula.
Hint: Complex notation is helful: use $\cos(n\psi) = \mathrm{Re}\,(\exp(i\psi))^n$.

e) What do you obtain for $r = 0$?

H7. *(7 points). Second proof of the maximum principle*
In this problem, we will prove the maximum principle without using the mean-value property. Assume U is a bounded, open domain in $\mathbb{R}^2$. We consider Poisson's equation

$$-\Delta u = f \text{ in } U.$$

a) Assume $-\Delta u = f < 0$ first, and suppose u has a maximum in some point $(x,y) \in \mathbb{R}^2$. What can you say about the first and second derivative of u? Use this to show that the maximum of u is attained on the boundary ∂U.

b) Next, show the same is true for $-\Delta u = f \leq 0$. Consider $v(x,y) := u(x,y) + \epsilon(x^2+y^2)$, where ϵ is a small constant. Show that a maximum principle for v holds. By letting ϵ tend to 0, prove that maximum of u is attained on the boundary ∂U.