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Homework Assignment #4

AM 125c

Engineering Mathematical Principles

H12. *(7 points). Maximum principle for the heat equation*

a) Suppose U is a bounded region in $\mathbb{R}^n$ $(n = 2, 3)$. Assume further $k > 0, T > 0$ and that u is a continuous solution of the heat equation

$$\frac{\partial u}{\partial t} - k\Delta u = 0 \text{ in } U \times (0, T)$$

with continuous derivatives up to order two in $U \times (0, T]$. Show that u attains its maximum either on U at $t = 0$ or on ∂U for $0 \leq t \leq T$. What does this imply if u is the temperature in some heat conducting body U?

b) Show that the inhomogeneous heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} - k\Delta u &= f \text{ in } U \times (0, T) \\ u(x, t) &= g(x, t) \text{ for } x \in \partial U, t \in (0, T), \\ u(x, 0) &= u_0(x) \text{ in } U \end{aligned}$$

has at most one solution with the smoothness properties in a).

H13. *(7 points). Well-posedness of initial-value problems*
Assume we prescribe initial conditions and boundary data for the following equations in such a way that they have a unique, smooth solution.

a) Show that the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \text{ on } (0, L) \times (0, \infty)$$

$(k > 0)$ with inhomogeneous boundary conditions and initial conditions is well-posed.

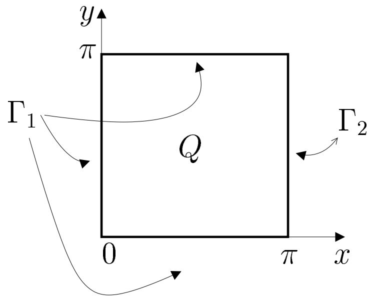
b) The equation

$$\frac{\partial u}{\partial t} = -k\frac{\partial^2 u}{\partial x^2} \text{ on } (0, L) \times (0, \infty)$$

$(k > 0)$ is often referred to as *backward heat equation*. Why? Show that this equation subject to $u(0, t) = u(L, t) = 0$ and $u(x, 0) = f(x)$ is not well-posed. *Hint:* What happens if you change the initial conditions $f(x)$ by adding function of the form $p(x) = \frac{1}{n}\sin\left(\frac{n\pi x}{L}\right)$?

H14. *(10 points).*
Let Q be the square in the figure on the right with $\partial Q = \Gamma_1 \cup \Gamma_2$. Consider the heat equation

$$\begin{aligned} u_t - \Delta u &= 0 \text{ in } Q, t > 0 \\ u &= 0 \text{ on } \Gamma_1, t > 0, \\ u &= 1 \text{ on } \Gamma_2, t > 0, \\ u &= 0 \text{ in } Q, t = 0. \end{aligned}$$



a) What is the equation satisfied by a stationary solution u_0 of the heat equation? Find the corresponding solution.
Hint: Use the Fourier series from **H10.**

b) Let u be the solution of the heat equation and define v by $u(x,y,t) = u_0(x,y) + v(x,y,t)$. Find the partial differential equation for v and the corresponding initial and boundary conditions. Determine v.
Hint: You may want to use the following formula:

$$\int_0^\pi \sinh(mx)\sin(nx)\mathrm{d}x = (-1)^{n+1}\frac{\sinh m\pi}{m^2+n^2}.$$

Remark. The decomposition seems a bit arbitrary, but it is an example of a useful technique: Essentially, the decomposition $u(x,y,t) = u_0(x,y) + v(x,y,t)$ gives a decomposition of the problem into two easier problems, one with nonvanishing boundary conditions without initial conditions and one with initial conditions and trivial boundary conditions.

c) Calculate $u_\infty(x,y) = \lim_{t\to\infty} u(x,y,t)$.

d) Find the maximum and minimum of u_∞.

H15. *(7 points). Energy principle for the heat equation*
Let U be a bounded region in $\mathbb{R}^n$. Consider the heat equation with homogeneous Dirichlet data:

$$\begin{aligned} u_t &= \Delta u \text{ in } U \\ u &= 0 \text{ on } \partial U. \end{aligned}$$

By multiplying the heat equation with an appropriate function, show that

$$\int_U u^2(t,x)\mathrm{d}x \le \int_U u^2(0,x)\mathrm{d}x.$$

Remark. If we define the *energy* $E(u)$ of u at time t as $E(u) := \int_U u^2(t,x)\mathrm{d}x$, this shows that the energy of the system is not increasing. Often, this is the first step to show existence of a solution.