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Homework Assignment #5

AM 125c

Engineering Mathematical Principles

H16. *(10 points).*

a) Solve the first order PDE

$$uu_x + u_y = 1, \qquad u(x,x) = \frac{x}{2} \text{ for } x \in \mathbb{R}.$$

Where is the solution well-defined? Plot the characteristic curves $\gamma(t) = (x(s,t), y(s,t))$ for several values s in the (x,y)-plane. What is remarkable?

b) Use the methods of characteristics to solve the first order PDE

$$yu_x + uu_y = 1, \qquad u(0,y) = 0 \text{ for } y > 0.$$

Find an implicit representation of the solution, $F(x, y, u(x,y)) = 0$ (it seems to be difficult to find an explicit formula for $u(x,y)$). Verify your solution by differentiating the representation.

H17. *(10 points). Characteristics and characteristic curves*

a) Solve

$$uu_x + u_y = 1$$

for the initial curve $f(s) = \frac{1}{2}s^2, g(s) = s, h(s) = s$ with the method of charcteristics. Is the curve a characteristic? Show that there are infinitely many smooth solutions of the given initial value problem: for every $w \in C^1(\mathbb{R})$, the equation

$$x = \frac{1}{2}u(x,y)^2 + w\left(u(x,y) - y\right)$$

defines a solution of the differential equation. Give a condition on w to guarantee that the initial curve is on the solution surface.

b) Investigate the same problem for the initial curve $f(s) = s^2, g(s) = 2s, h(s) = s$. For $x > y^2/4$, solve x and y for s and t to obtain a solution. Show the solution surface contains the initial curve. Is the solution differentiable along the curve? Is the initial curve a characteristic?

H18. *(10 points). The eikonal equation*
The equation

$$u_x^2 + u_y^2 = 1$$

arises in geometric optics, it is called *eikonal equation*. The level lines of a solution u can be interpreted as wave fronts; they mark the location to which the light has spread (the velocity is normalized to be 1).

a) Discuss solvability (existence and uniqueness) of the initial value problem $u(x, x) = ax$ for $a \in \mathbb{R}$.

b) Describe the Monge cone in an arbitrary point (x_0, y_0, z_0) of the solution surface.

c) Show a family of solutions is given by

$$u_s(x, y) = x\cos(s) + y\sin(s).$$

Compute the envelope of this family to obtain another solution. What is the physical interpretation of that solution?

H19. *(10 points).* Use the methods of characteristics to solve the first order PDE

$$\begin{aligned}(1 + ku)u_x + u_y &= 0, \qquad x \in \mathbb{R}, y \geq 0,\\ u(x, 0) &= u_0(x), \qquad x \in \mathbb{R},\end{aligned}$$

where $k > 0$ and the function u_0 is given by

$$u_0(x) = \begin{cases} 1 & \text{if } x \leq 0,\\ 1 - x & \text{if } x \in [0, 1],\\ 0 & \text{if } x \geq 1,\end{cases}$$

a) What are the characteristic equations and their solutions? Sketch the characteristic curves $\gamma_s(t) = (x(s, t), y(s, t))$ in the x, y-plane. Verify that the solution u is constant along the solutions γ_s.

b) Let y^* be the smallest value for y for which some of the curves γ_s intersect. Sketch the solution u for $y = 0, y^*/2, y^*$. How would the solutions look like for $y > y^*$? Is this "solution" still a function of x and y?