Johannes Zimmer

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Homework Assignment #6

AM 125c
Engineering Mathematical Principles

H20. *(10 points). Burgers' equation*

a) Find the entropy solution of Burgers' equation with initial data

$$u(x,0) = \begin{cases} 2 & \text{if } x < 0 \\ 1 & \text{if } 0 < x < 2 \\ 0 & \text{if } x > 2 \end{cases}$$

Sketch the characteristics of the solution. Make sure your sketch shows the behavior for all $y > 0$.

b) Find a weak solution of Burgers' equation with initial data

$$u(x,0) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

that has both a rarefaction wave and a rarefaction fan. Is it an entropy solution?

H21. *(7 points).*

a) Prove that any C^1 solution of

$$u_t + f(u)_x = 0$$

for continuously differentiable f is constant along the characteristics, and that the characteristic curves are straight lines.

b) Consider again Burgers' equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0$$

and multiply the equation by u to obtain an "equivalent" equation. Show that the new equation can be written as a conservation law. Choose some initial data to show that the shock speeds given by the Rankine-Hugoniot condition do not necessarily agree. So what can you say about discontinuous solutions of "equivalent" equations?

H22. *(10 points). Numerical schemes for a linear scalar hyperbolic equation*
In this problem, we investigate numerical methods for the initial value problem

$$\begin{aligned} u_t + cu_x &= 0, \\ u(x,0) &= f(x), \end{aligned}$$

where $c > 0$ is constant. First, introduce a grid in the (x,t) plane, using $x_j = j\Delta x$ and $t_k = k\Delta t$ ($j \in \mathbb{Z}, k \in \mathbb{N}$). We seek approximations $u_{j,k} \approx u(x_j, t_k)$.

a) Using *forward difference quotients* in u_t and u_x, a natural approximation is

$$u_{j,k+1} = u_{j,k} - c\frac{\Delta t}{\Delta x}\left(u_{j+1,k} - u_{j,k}\right).$$

Describe how to compute the solution iteratively. Consider the initial data

$$f(x) = \begin{cases} 0 & \text{if } x < -1 \\ x+1 & \text{if } -1 \leq x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

Compute the exact solution. Does the approximation converge to the exact solution as $\Delta t, \Delta x \to 0$? Explain the reason for this.

b) Now we use *forward differences* in u_t and *backward differences* in u_x:

$$u_{j,k+1} = u_{j,k} - c\frac{\Delta t}{\Delta x}\left(u_{j,k} - u_{j-1,k}\right).$$

(In gas dynamics, this is called an *upwind scheme.*) Explain why this discretization is likely to give a better approximation of the initial value problem in a).

c) The number $\rho := \frac{\Delta x}{\Delta t}$ is the speed of propagation of the numerical scheme. The *Courant-Friedrichs-Levy (CFL) condition* requires that

$$\frac{c}{\rho} \leq 1.$$

Explain why a violation of the CFL condition will usually give a wrong approximation.

H23. *(10 points). Traffic flow*
A simple model for the traffic in a one-lane road without on and off ramps: let $0 \leq \rho \leq \rho_{\max}$ be the traffic density (number of cars divided by unit distance of the road). Consider the PDE

$$\rho_t + F(\rho)_x = 0.$$

For simplicity, we assume $F(\rho) := R\rho\,(\rho_{\max} - \rho)$, where R is constant and $v := R\rho$ is the velocity.

a) Why is this PDE with the given flux F a simple model for traffic flow?

b) Assume we are solving the conservation law with initial condition

$$\rho(x,0) = \begin{cases} \rho_l & \text{if } x < 0 \\ \rho_r & \text{if } x > 0 \end{cases},$$

with $0 \leq \rho_l, \rho_r \leq \rho_{\max}$. Under what conditions can we expect an entropy solution (a shock wave; a rarefaction wave)? (Optional: does this reflect your own experience?) Determine the shock speed.

c) Let $\rho_l = \frac{1}{2}\rho_{\max}$ and $\rho_r = \rho_{\max}$. Find the solution ρ.

d) Let $\rho_l = \rho_{\max}$ and $\rho_r = \frac{1}{2}\rho_{\max}$. Find the solution ρ.