Johannes Zimmer

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Homework Assignment #7

AM 125c
Engineering Mathematical Principles

H24. *(10 points).*

a) Show that

$$\varphi(x) = \begin{cases} \exp\left(\frac{1}{x^2-1}\right) & \text{if } x \in (-1,1) \\ 0 & \text{else} \end{cases}$$

is infinitely differentiable. Prove rigorously that φ is continuously differentiable and indicate how one could prove that all derivatives of φ are continuous.

b) Prove the *Fundamental Lemma of the Calculus of Variations* (also known as Lemma of du Bois-Reymond): Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and

$$\int_{-\infty}^{\infty} f(x)\varphi(x)\mathrm{d}x = 0$$

for all $\varphi \in C_c^\infty(\mathbb{R})$. Then $f \equiv 0$.

H25. *(10 points). The Cole-Hopf transformation*
The viscous Burgers' equation is

$$u_t + uu_x = \epsilon u_{xx}.$$

(This viscous regularization of Burgers' equation was proposed by Burgers in 1948 as a model of turbulence; it is in the spirit of viscosity solutions). Hopf and Cole independently showed that this and other nonlinear problems can be transformed to a linear initial value problem for the heat equation:

a) The *Cole-Hopf transformation* reduces a parabolic PDE with quadratic nonlinearity

$$\begin{aligned} u_t + b\|Du\|_2^2 &= a\Delta u \text{ in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) &= f(x) \end{aligned}$$

to the heat equation

$$\begin{aligned} w_t &= a\Delta w \\ w(x,0) &= g(x). \end{aligned}$$

Show that this transformation can be accomplished by setting $w := \exp(-bu/a)$: start with a general transformation $w = \varphi(u)$ and show that, up to constants, $\varphi(u) = \exp(-bu/a)$. What is the relation between f and g?

b) The Cole-Hopf transformation can also be applied to the viscous Burgers' equation

$$\begin{aligned} u_t + uu_x &= \epsilon u_{xx} \\ u(x,0) &= f(x). \end{aligned}$$

To see this, determine the PDE for $v(x,t) := \int_{-\infty}^{x} u(y,t)\mathrm{d}y$. Compute the solution of the initial-value problem.

H26. *(7 points). Entropy/entropy flux pairs*
Suppose u is a piecewise continuous solution of

$$u_t + f(u)_x = 0$$

and satisfies the entopy condition (in the sense of entropy/entropy flux pairs). Suppose further that u has a jump discontinuity along a shock with shock speed $\dot{s}$. Show that

$$\dot{s}[\![\eta(u)]\!] - [\![\psi(u)]\!] \geq 0$$

for every entropy/entropy flux pair η, ψ.

H27. *(5 points). Characteristics for second order PDEs*
Consider the *Tricomi equation*

$$u_{yy} - yu_{xx} = 0$$

in $\mathbb{R}^2$. Determine the type of the equation for an arbitrary point $(x,y) \in \mathbb{R}^2$. Find the normal form of the hyperbolic case.