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AM 125c

Engineering Mathematical Principles

Final Exam

120 minutes. Open notes and textbooks. There are 5 problems (40 points) on the exam. Please show all your work: we cannot give credits for unsupported results.

1. *(8 points). Heat conduction in a wall*
Suppose a wall of thickness L separates a room at temperature u_0 from another room at temperature u_L. At $t = 0$, the temperature inside the wall is given by a function $f(x)$; the diffusivity c is constant and positive:

$$\begin{aligned} u_t &= c u_{xx} \text{ for } 0 < x < L \text{ and } t > 0, \\ u(0,t) &= u_0, \\ u(L,t) &= u_L, \\ u(x,0) &= f(x) \text{ for } 0 < x < L. \end{aligned}$$

a) Find a solution u^* of the corresponding stationary problem.

b) Let u be the solution of the heat equation and define v by $u(x,t) = u^*(x) + v(x,t)$. Find the partial differential equation for v and the corresponding initial and boundary conditions. Use separation of variables to determine v (explain your choice of the sign; carry out the procedure only for this choice). Write v as a superposition (no need to determine the coefficients as functions of the Fourier coefficients of $f - u^*$).

2. *(8 points).* Solve the nonlinear first order PDE

$$u_x^3 - u_y = 0, \qquad u(x,0) = 2x^{3/2}$$

for $x > 0$.

3. *(9 points). Burgers' equation with a pulse as initial data*
Consider Burgers' equation in $U := \mathbb{R} \times (0, \infty)$,

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0$$

with initial data

$$u(x,0) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x > 1 \end{cases}.$$

Find an entropy solution. Sketch the characteristics along which the solution is constant. Make sure the Rankine-Hugoniot condition is satsified whenever discontinuities intersect.

4. *(9 points)*. Let $U := \{(x,y) \mid (x,y) \in \mathbb{R}^2, |x| < 1, |y| < 1\}$. Suppose $u \in C^\infty(U,\mathbb{R}) \cap C^1(\overline{U},\mathbb{R})$ is a solution of

$$\begin{aligned} \Delta u &= -1 \text{ in } U & (1)\\ u &= 0 \text{ on } \partial U. & (2) \end{aligned}$$

a) Use the symmetry operations $x \mapsto -x$ and $y \mapsto -y$ to generate three "additional" solutions u_1, u_2, u_3 from u (consider, for example, $u_1(x,y) := u(-x,y)$). What can you conclude for u?

b) Show that $v_1 := \frac{\partial}{\partial x}u$ and $v_2 := \frac{\partial}{\partial y}u$ solve

$$\Delta v = 0 \text{ in } U.$$

Show that the signs of v_1 and v_2 are constant on each of the four line segments of ∂U; determine the signs.
Hint: Use that the Poisson equation $\Delta u = f < 0$ satisfies a strong minimum principle: the minimum of u is attained only on the boundary.

c) Show that

$$\big\{(x,y) \mid (x,y) \in U, v_1(x,y) = 0, v_2(x,y) = 0\big\} = \{(0,0)\}.$$

Hint: For v_1, apply the maximum principle, for example, to the domain

$$V_1^- := \{(x,y) \mid (x,y) \in \mathbb{R}^2, -1 < x < 0, |y| < 1\}.$$

d) Conclude that

$$\sup_U u(x,y) = u(0,0).$$

5. *(6 points)*. Consider the conservation law

$$u_t + f(u)_x = 0.$$

Suppose that $f'' > 0$ and that the initial data u_0 is not monotonically increasing. Prove that there is no C^1 solution defined in the upper half plane $(x,t) \in (-\infty,\infty) \times [0,\infty)$.